Defn An isomorphism (Greek: "same form") from a vector space V to a vector space W is an invertible linear transformation $V \to W$. We say that V and W are isomorphic if there exists an isomorphism between them.

Reminder (invertible functions): Let $f: X \to Y$ be a function. TFAE:

- f is invertible, i.e. $\exists f^{-1}: Y \to X$ s.t. $f^{-1}(f(x)) = x \ \forall x \in X$ and $f(f^{-1}(y)) = y \ \forall y \in Y$.
- f is one-to-one and onto.
  injective surjective

If f is invertible, the function f^{-1} is uniquely determined.

Thm: If $T: V \to W$ is an isomorphism of vector spaces, then so is $T^{-1}: W \to V$.

This is a fancy version of the following result:

If A is an invertible $n \times n$ matrix, then the function

$$\mathbb{R}^n \to \mathbb{R}^n, \ x \mapsto Ax$$

is invertible. Its inverse is $\mathbb{R}^n \to \mathbb{R}^n$, $y \mapsto A^{-1}y$.

inverse matrix of A.

Ex (isomorphisms):

- For any vector space V, the identity map

$$id_V : V \to V, \; x \mapsto x$$

is an isomorphism. (Hence: every vector space is isomorphic to itself)

- Given any basis $\mathcal{B} = (b_1, \ldots, b_n)$ of V, the coordinate mapping $V \to \mathbb{R}^n$, $x \mapsto [x]_{\mathcal{B}}$ is an isomorphism by the Thm on p. 42.

Coordinate mappings for $\mathbb{R}^n$:

Let $\mathcal{B} = (b_1, \ldots, b_n)$ be a basis of $\mathbb{R}^n$.

(WARNING: At this point, we don't yet know whether all bases of $\mathbb{R}^n$ consist of n vectors.)

Then $T : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto [x]_{\mathcal{B}}$ and its inverse $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$, $[x]_{\mathcal{B}} \mapsto x$ are both linear transformations from $\mathbb{R}^n$ to itself.

By definition, $T(x) = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \iff x = c_1 b_1 + \ldots + c_n b_n$

$$\iff T^{-1} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = x.$$

We know: linear transformations $\mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto Ax$

$$\updownarrow \qquad\qquad \updownarrow$$

$n \times n$ matrices $\qquad A$

What are the matrices corresponding to T and T^{-1}?

This is easier for T^{-1}; Indeed,

$$T^{-1}\left(\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) = b_i$$

in position i

so the matrix of T^{-1} is $A := [b_1 \cdots b_n]$.

One can then deduce that A^{-1} is the matrix of T.

It turns out that invertible $n \times n$ matrices and basis of $\mathbb{R}^n$

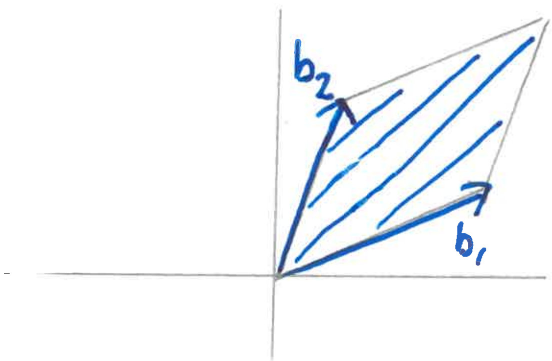
are in 1-1 correspondence via

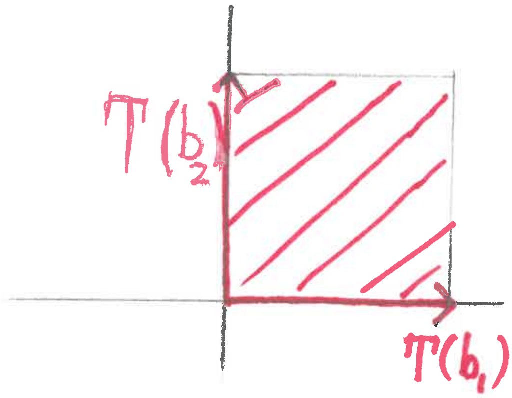
$$[b_1 \cdots b_n] \longleftrightarrow (b_1, \dots, b_n).$$

Graphical interpretation

Let $\mathcal{B} = (b_1, b_2)$ be a basis of $\mathbb{R}^2$. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$, $x \mapsto [x]_{\mathcal{B}}$ be the associated coordinate mapping — an isomorphism. Then

$$T(b_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad T(b_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$





Each basis of $\mathbb{R}^2$ defines a parallelogram. The corresponding coordinate mapping "stretches", "rotates", and possibly "reflects" it into a square.

Computer graphics

- A scene is an arrangement of objects in $\mathbb{R}^3$.
- "Rotations about the origin" and "reflections along subspaces" are isomorphisms from $\mathbb{R}^3$ onto itself (so-called automorphisms). These are related to so-called orthogonal matrices. → LATER
- Our 3D scene is projected onto a "flat 2D screen". This is another linear transformation, e.g. $\mathbb{R}^3 \to \mathbb{R}^2$, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \end{bmatrix}$.
- More work: by going to $\mathbb{R}^4$, "linear motions" in $\mathbb{R}^3$ become linear transformations