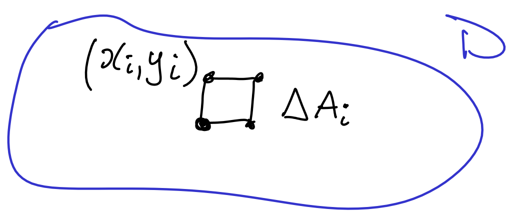
Lecture 9 (W5L1) Applications

1/ Centre of Mass of an object



D is a lamina (2d solid object)

D is has mass m

density @ (x,y) is $\rho(x,y) = \lim \frac{\Delta m}{\Delta A}$

[ρ = rho]

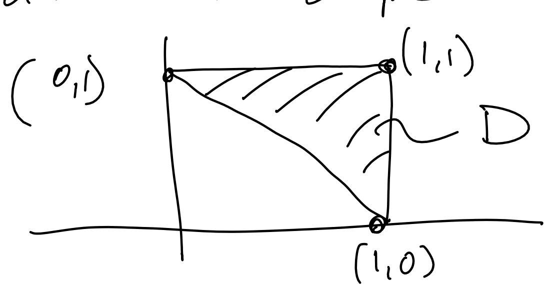
• Break D into small rectangles, labelled (x_i, y_i)

- mass of a rectangle is m_i
where $m_i = \rho(x_i,y_i)\Delta A_i$

Total mass of D is then $\sum_i \rho(x_i,y_i)\,\Delta A_i$

In the limiting case $\boxed{m = \iint_D \rho(x,y)\,dA}$

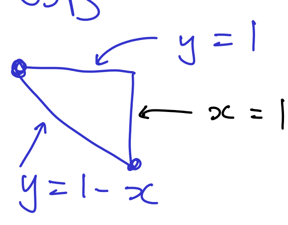
Ex 1: Suppose the density of a lamina is given by $\rho(x,y) = xy$ [Units kg/m³ or equivalent.]
and its shape is as shown:



Find its mass.

Sol^n: $m = \iint_D xy \, dA$

[Find limits of integ.



y ranges from $1-x$ to 1
x " 0 to 1]

[Exercise: repeat this analysis treating D as a type II region]

$$= \int_{x=0}^{1} \int_{y=1-x}^{1} xy \, dy \, dx$$

Inner: $\displaystyle\int_{y=1-x}^{1} xy\,dy = \left. \frac{xy^2}{2} \right]_{1-x}^{1} = \frac{x}{2}(1)^2 - \frac{x}{2}(1-x)^2$

$$= \frac{x}{2} - \frac{x}{2}(1-2x+x^2)$$

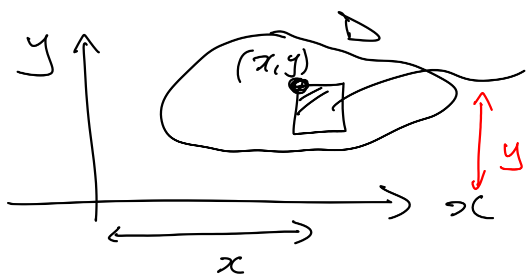
$$= 2x \cdot \frac{x}{2} - x^2 \cdot \frac{x}{2}$$

Outer: $\displaystyle\int_0^1 x^2 - \frac{x^3}{2}\,dx = \left[\frac{x^3}{3} - \frac{1}{2}\frac{x^4}{4} \right]_0^1$

$$= \left(\frac{1}{3} - \frac{1}{8}\right) - (0) = \frac{5}{24} \text{ (kg)}$$

1st Moments about x-axis and y-axis.

M_x = moment about x-axis



mass of rectangle

x-distance from the y-axis

Distance from x-axis is y.

For each little rectangle, its distance from x-axis is y

$$M_x = \iint_D \underbrace{\rho(x,y)\,dA}_{} \cdot y$$

mass of tiny rectangle × its dist from x-axis

$$\boxed{M_x = \iint_D y\,\rho(x,y)\,dA}$$

Similarly, $\boxed{M_y = \iint_D x\,\rho(x,y)\,dA}$

We define the centre of mass (c.o.m.) as $(\bar{x}, \bar{y})$

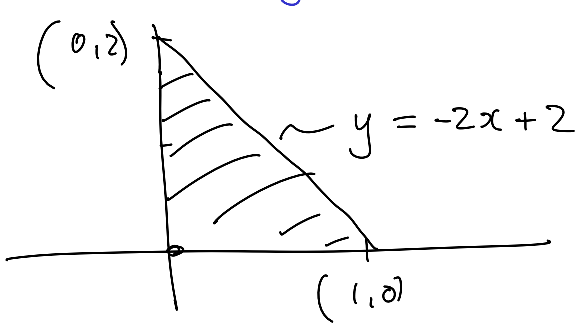
such that
$$m\bar{x} = M_y$$
$$m\bar{y} = M_x$$

thus $\bar{x} = \frac{1}{m} M_y$ and $\bar{y} = \frac{1}{m} M_x$

ie $\boxed{\bar{x} = \frac{1}{m} \iint_D x \rho(x,y)\, dA}$ and $\boxed{\bar{y} = \frac{1}{m} \iint_D y \rho(x,y)\, dA}$

Ex 2 Find the mass and c.o.m. of a triangular lamina with vertices $(0,0)$, $(1,0)$ and $(0,2)$ if $\rho = 1 + 3x + y$.

Sol^n:



$$m = \iint_D \rho \, dA$$

$$m = \int_0^1 \int_{y=0}^{-2x+2} 1 + 3x + y \, dy\, dx$$

$$= \dots = \frac{8}{3}$$

Then $\bar{x} = \frac{1}{m} \iint_D x \rho(x,y)\, dA = \frac{1}{m} \int_{x=0}^{1} \int_{y=0}^{-2x+2} x + 3x^2 + xy \, dy\, dx$

Similarly for $\bar{y}$.