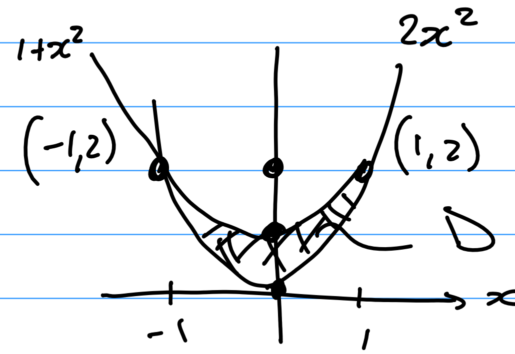
W3L3 Ex of $\iint$ over Type I region (Stewart)

Ex $f(x,y) = x + 2y$ ; $x$ between $[-1, 1]$

and $2x^2 \le y \le 1 + x^2$




Sol$^n$ :

Note D is enclosed between $g_2 = 1 + x^2$ and $g_1 = 2x^2$

Let's compute the points of intersection : $1 + x^2 = 2x^2$

$\Rightarrow 1 = x^2$

$\Rightarrow x = \pm 1 \Rightarrow y = 2x^2 \Rightarrow y = 2$

$\therefore (-1,2)$ and $(1,2)$ are our limiting points of D.

Note D is Type I

From Monday's notes : $\iint_D f \, dA = \int_{x=-1}^{1} \int_{y=2x^2}^{y=x^2+1} f(x,y) \, dy \, dx$

Inner integral : $\int_{2x^2}^{x^2+1} x + 2y \, dy = \left[ xy + \frac{2y^2}{2} \right]_{2x^2}^{x^2+1}$

$$= \left( x(x^2+1) + (x^2+1)^2 \right) - \left( x(2x^2) + (2x^2)^2 \right)$$
$$= \left( x^3 + x + x^4 + 2x^2 + 1 \right) - \left( 2x^3 + 4x^4 \right)$$
$$= -3x^4 - x^3 + 2x^2 + x + 1$$

Outer int : $\int_{-1}^{1} \left( -3x^4 - x^3 + 2x^2 + x + 1 \right) dx$

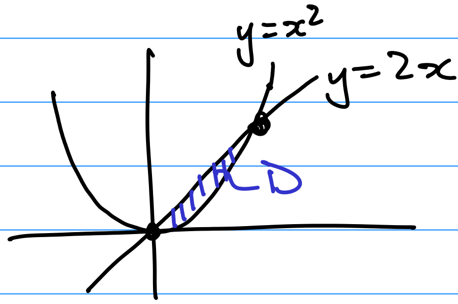
$$= \left[ -\frac{3x^5}{5} - \frac{x^4}{4} + \frac{2x^3}{3} + \frac{x^2}{2} + x \right]_{-1}^{1}$$

$$= \left(-\frac{3}{5} - \frac{1}{4} + \frac{2}{3} + \frac{1}{2} + 1\right) - \left(\frac{3}{5} - \frac{1}{4} - \frac{2}{3} + \frac{1}{2} - 1\right)$$

$$= -6/5 + 4/3 + 2 = \frac{-18 + 20 + 30}{15} = \frac{32}{15} \approx 2\,2/15$$

**Ex** Find volume of the solid defined by the region under $z = x^2 + y^2$ (circular paraboloid) and above the region $D$ on the $x$-$y$ plane which is enclosed between $y = 2x$ and $y = x^2$.

**Sol^n:**




Let $2x = x^2$

$\Rightarrow \quad x^2 - 2x = 0$

$\Rightarrow \quad x(x-2) = 0$

$\Rightarrow x = 0 \quad$ or $\quad x = 2$

$\therefore y = 2x \Rightarrow y = 0 \quad$ or $\quad y = 4$

So $(0,0)$ and $(2,4)$ are points of int. on $D$.

These give us limits of integ.

Note $D$ is Type I [It's also Type II - exercise]

For each $(x,y) \in D$, $y$ is between $x^2$ and $2x$

i.e. $\quad x^2 \le y \le 2x$

and $\quad 0 \le x \le 2.$

✓ So $$\iint_D f \, dA = \int_0^2 \int_{y=x^2}^{2x} x^2 + y^2 \, dy \, dx$$

Inner: $$\int_{x^2}^{2x} x^2 + y^2 \, dy = \left[x^2 y + \frac{y^3}{3}\right]_{x^2}^{2x}$$

$$= \left(x^2(2x) + \frac{8x^3}{3}\right) - \left(x^2(x^2) + \frac{x^6}{3}\right)$$

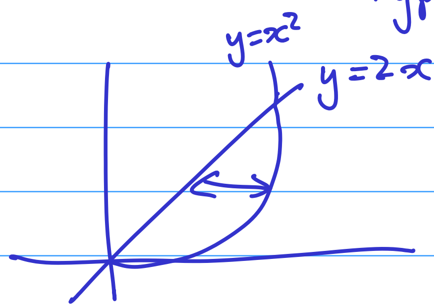
$$= 2x^3 + \frac{8x^3}{3} - \left( x^4 + \frac{x^6}{3} \right)$$

Outer: $$\int_0^2 \frac{14x^3}{3} - x^4 - \frac{x^6}{3} \, dx$$

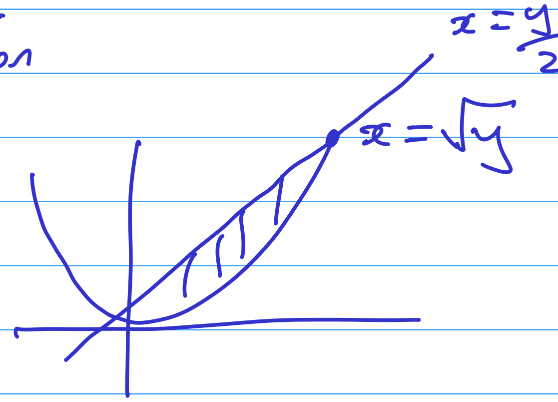
$$= \ldots \quad \text{[exercise]}$$

$$= \frac{216}{35} \approx 6.17$$

Exercise: Also a Type II region




$\longleftrightarrow$




Can do...
$$\int_{y=0}^{4} \int_{x=y/2}^{\sqrt{y}} x^2 + y^2 \, dx \, dy$$

As exercise!