W3L1

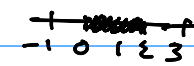
Double Integrals as volumes } see intro to Ch. 15 in Stewart.
Riemann sums

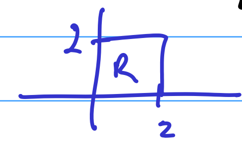
Ex

Let $f(x,y) = 16 - x^2 - 2y^2$

Find $\iint_R f(x,y)\, dA$

$[0,2]$





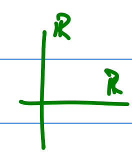
$R = [0,2] \times [0,2]$

$\mathbb{R}$ ——

$\mathbb{R}$ ——

$\mathbb{R} \times \mathbb{R}$

$\mathbb{R}^2$



Cartesian plane

$\times$ = Cartesian product

Ans: $\displaystyle\int_{y=0}^{2}\int_{x=0}^{2} 16 - x^2 - 2y^2 \, dx\, dy$ = Volume

Inner $\displaystyle\int_0^2 16 - x^2 - 2y^2 \, dx$ =

$$= \left[16x - \frac{x^3}{3} - 2y^2 x\right]_0^2 = \left(32 - \frac{8}{3} - 4y^2\right) - (0)$$

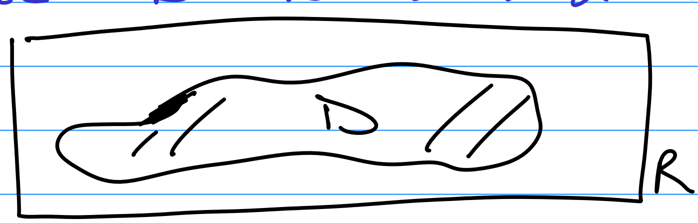
Outer : $\displaystyle\int_0^2 \left(\frac{88}{3} - 4y^2\right) dy$

$$= \frac{88}{3}y - \frac{4y^3}{3}\Big]_0^2 = \frac{88}{3}(2) - \frac{4(8)}{3} - (0 - 0)$$

$$= \frac{144}{3} = 48$$

Integration over general regions

- Suppose D is bounded ⇒ D can be enclosed in some rectangle R.



If $f(x,y)$ is defined over D

we introduce $F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D \\ 0 & \text{if } (x,y) \in R \setminus D \end{cases}$

So $F(x,y)$ is defined over R.

So $\iint_R F(x,y)\,dA$ makes sense.

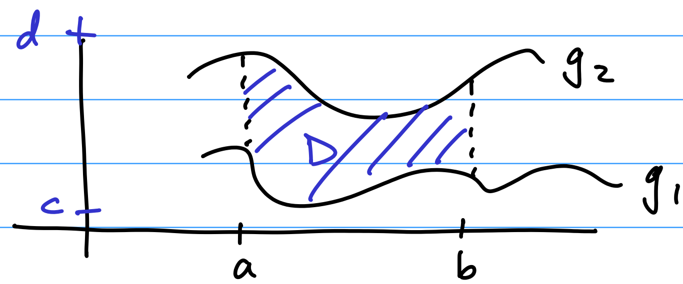
We extend this to $\iint_D f(x,y)\,dA$

Type I Regions (in $\mathbb{R}^2$)

Say D is of type I if we have functions $g_1(x)$ and $g_2(x)$ such that

$$g_1(x) \le y \le g_2(x) \quad \text{for } (x,y) \in D.$$

e.g.



Then $\iint_R F(x,y)\,dA = \int_a^b \int_c^d F(x,y)\,dy\,dx$

Now $\int_c^d F(x,y)\,dy = \int_{g_1(x)}^{g_2(x)} F(x,y)\,dy$

since F is 0 below g_1 and above g_2

$= \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy$

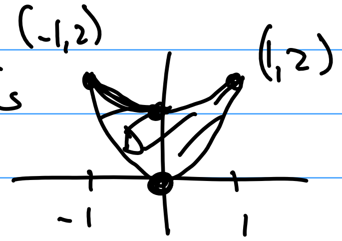
since f and F coincide between g_1 & g_2

and hence $\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$

This is how we apply our known techniques (based on rectangles) to compute integrals over type I regions D.

Ex: Exercise

Compute $\iint_D (x + 2y)\, dA$ where D is



[D bounded between $y = 2x^2$ and $y = x^2+1$]