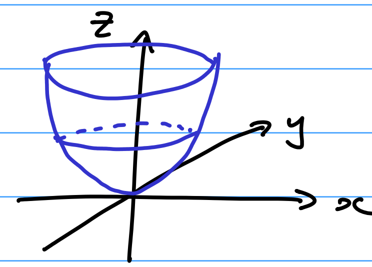
WIL2

Recall $f: \mathbb{R}^2 \to \mathbb{R}$ graph will describe a surface in $\mathbb{R}^3$

e.g. $f(x,y) = x^2+y^2$

$z = x^2 + y^2$

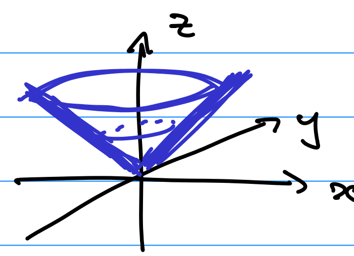


circular paraboloid

e.g. $g(x,y) = \sqrt{x^2+y^2}$

$z = \sqrt{x^2+y^2}$

cone



check shape by sketching contours.

e.g. $h(x,y) = $ polynomial

$= x^3 + 4x^2y + 3xy^3 - 5$

(sketch with software)

Recap of Partial diff^n:

e.g. Compute $\frac{\partial h}{\partial x} = h_x$ [Treat y as constant]

$$\frac{\partial h}{\partial x} = 3x^2 + 4(2x)y + 3y^3(1) - 0$$

$$= 3x^2 + 8xy + 3y^3$$

$$\frac{\partial h}{\partial y} = x^3 + 4x^2 + 6xy^2 - 0 \quad ✗$$

$$4x^2 + 9xy^2 \quad ✓✓$$

$x^3 + \cancel{4x^2 + 9xy^2}$ $\}$

$$h(x,y) = x^3 + 4x^2y + 3xy^3 - 5$$
$$h_y = 0 + 4x^2(1) + 3x(3y^2) - 0$$

<u>Recap</u> 1-var Integration

$$\int Kx^n\,dx = K\frac{x^{n+1}}{n+1} + C$$

K = constant

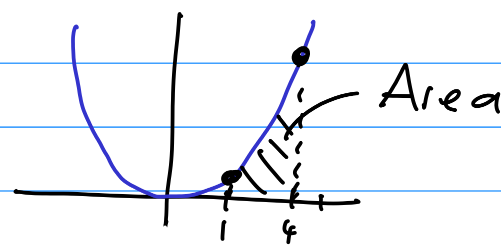
$$\int K\,dx = Kx + C$$

e.g. $\displaystyle\int 7x^2+4\,dx = \frac{7x^3}{3} + 4x + C$

e.g. $\displaystyle\int 2y^2\,dy = \frac{2y^3}{3} + C$

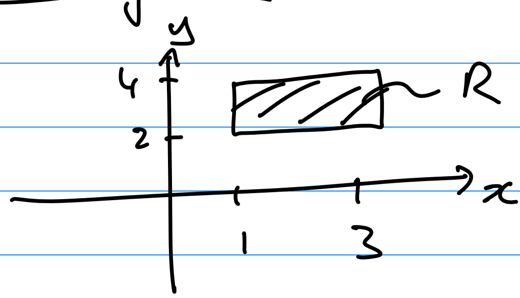
Recall $\displaystyle\int_1^4 x^2\,dx = \frac{x^3}{3}\Big]_1^4 = \frac{4^3}{3} - \frac{1^3}{3} = \frac{63}{3} = 21$

We interpret definite integrals:



<u>Double Integrals</u>: Let R = rectangular region in plane

e.g.



eg. $R = \{(x,y)\in\mathbb{R}^2 : 1\le x\le 3,\ 2\le y\le 4\}$

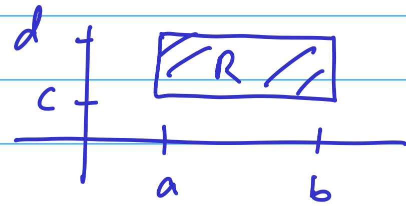
or $R = \{(x,y)\in\mathbb{R}^2 : x\in[1,3],\ y\in[2,4]\}$

or $R = [1,3]\times[2,4]$

def^n: $\iint_R f(x,y)\,dA$ = double integral of f over rectangular region R

~ interpret as volume enclosed between f and x-y plane [assuming $f \geq 0$]

If



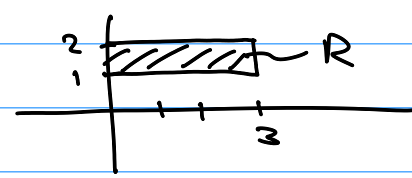
ie $R = [a,b] \times [c,d]$

$$= \int_{x=a}^{b} \int_{y=c}^{d} f(x,y)\,dy\,dx$$

$$= (\text{Fubini}) = \int_{y=c}^{d} \int_{x=a}^{b} f(x,y)\,dx\,dy$$

Ex (Stewart) Compute $\int_0^3 \int_1^2 x^2 y\,dy\,dx$

Ans: Note implicit nesting here; here $1 \leq y \leq 2$ and $0 \leq x \leq 3$; thus



This is an iterated integral; compute $\int_1^2 x^2 y\,dy$ first

$dy \Rightarrow$ Treat x as a constant & integrate wrt y

$$= x^2 \frac{y^2}{2} \Big]_{y=1}^{2} = \frac{x^2}{2}(2)^2 - \frac{x^2}{2}(1)^2$$

$$= 2x^2 - \frac{x^2}{2} = \frac{3x^2}{2}$$

Now compute $\int_0^3 \left(\qquad \right) dx$; outer integral

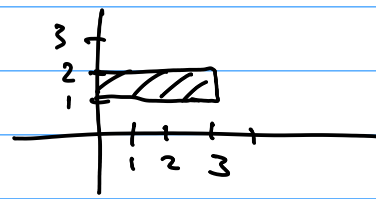
$$= \int_0^3 \frac{3x^2}{2}\, dx$$

$$= \frac{3}{2} \cdot \frac{x^3}{3} \Big]_0^3 = \frac{x^3}{2} \Big]_0^3 = \frac{3^3}{2} - \frac{0^3}{2} = \frac{27}{2}$$

Note, in the above, the inner integral $\int_1^2 x^2 y\, dy$ evaluates to give (eventually) a function of x

Write $A(x) = \int_1^2 x^2 y\, dy$

Note 2: R is implicit here,



$$\int_0^3 \int_1^2 x^2 y\, dy\, dx$$

y ranges from 1 to 2
x ranges from 0 to 3

Can switch the order of integration!
Same volume $= \int_1^2 \int_0^3 x^2 y\, dx\, dy$

Exercise: Compute $\int_1^2 \int_0^3 x^2 y\, dx\, dy$.

W222:
Ans: $\int_0^3 x^2 y\, dx$ = [hold y const] $= y \cdot \frac{x^3}{3} \Big]_0^3$

$$= y \frac{x^3}{3} \Big]_{x=0}^3$$

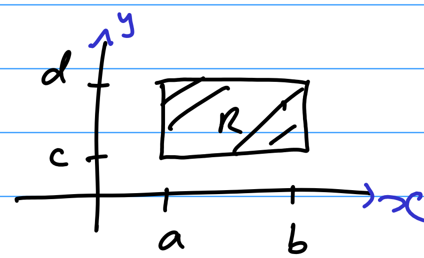
$$= y(9) - y(0) = 9y$$

Outer Int: $\displaystyle\int_1^2 9y\, dy = \frac{9y^2}{2}\Bigg]_1^2$

$$= \frac{9(4)}{2} - \frac{9(1)}{2} = \frac{27}{2}$$

as before.

<u>Note</u>: In general, $A(x) := \int_c^d f(x,y)\, dy$



here is the area of cross-section at constant value x in a plane which is parallel to the yz-plane.

Then, $\displaystyle\int_a^b \int_c^d f(x,y)\, dy\, dx$

$\displaystyle= \int_a^b A(x)\, dx$ is the volume of the region between f and R.

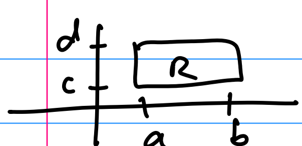
(we suppose $f \geq 0$ for simplicity)

Also, if $B(y) := \int_a^b f(x,y)\, dx$ then

$B(y)$ is the area of cross-section of constant y-value in a plane parallel to the xz plane and then

$\displaystyle\int_c^d B(y)\, dy$ = volume of solid region

This is a heuristic of Fubini's Theorem:



: $\displaystyle\int_a^b \int_c^d f(x,y)\, dy\, dx = \int_c^d \int_a^b f(x,y)\, dx\, dy$

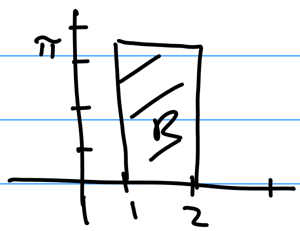
$$= \iint_R f(x,y)\,dA$$

= volume of region enclosed between f and R (f positive)

Example Given $R = [1,2] \times [0,\pi]$

Compute $\iint_R y \sin(xy)\,dA$.

Solution:



$$\text{Vol} = \int_0^{\pi}\int_1^2 y\sin(xy)\,dx\,dy$$

or $$\text{Vol.} = \int_1^2\int_0^{\pi} y\sin(xy)\,dy\,dx$$

Can compute (exercise) $\int_0^{\pi}\int_1^2 y \sin xy\,dx\,dy = 0$

[4 lines]

or other one [2 pages!!]

$$\int_1^2\int_0^{\pi} y \sin xy\,dy\,dx$$

So $\int_0^{\pi} y \sin xy\,dy$ ... use int. by parts

$u = y$ $\quad dv = \sin xy\,dy$

$du = dy$ $\quad v = -\frac{\cos xy}{x}$

$$= -\frac{y}{x}\cos xy\Big]_0^{\pi} - \int_0^{\pi} -\frac{\cos xy}{x}\,dy$$

$\int u\,dv = uv - \int v\,du$

$$= -\frac{y}{x}\cos xy\Big]_{y=0}^{\pi} + \underbrace{\int_0^{\pi}\frac{\cos xy}{x}\,dy}_{\frac{1}{x}\cdot\frac{\sin xy}{x}\Big]_0^{\pi}}$$

[continue as exercise!]