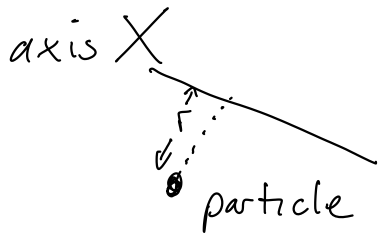
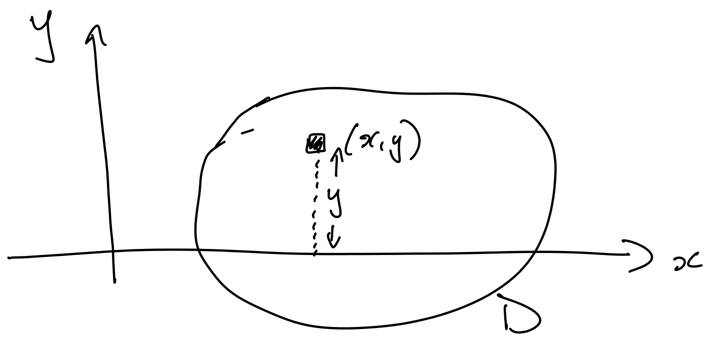
Lecture 11 - Applications (Moment of Inertia)
(2nd moment)



I_X denotes moment of inertia about X

$I_x = mr^2$



D lamina

• distance from (x,y) to the x-axis is y.

$$\boxed{I_x = \iint_D y^2 \rho(x,y)\, dA}$$

(moment of inertia about X axis)

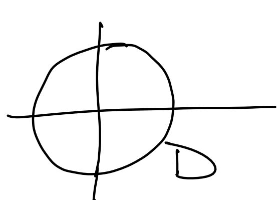
Also $$\boxed{I_y = \iint_D x^2 \rho(x,y)\, dA}$$

. We also study I_0 (moment of inertia about origin)

$$\boxed{I_0 = \iint_D (x^2+y^2)\rho(x,y)\, dA}$$

• Note that $I_0 = I_x + I_y$

Example: Find I_x, I_y and I_0 for a homogeneous disc D with $\rho(x,y) = \rho$.
[Centre is (0,0) and radius is a.]



D: $x^2+y^2=a^2$. Use polar form.

ie $0 \le \theta \le 2\pi$

$0 \le r \le a$

Given $I_0 = \iint_D (x^2+y^2)\rho \, dA$; $x^2+y^2=r^2$

$= \iint_D r^2 \cdot \rho \cdot r \, dr \, d\theta$; $dA = r\,dr\,d\theta$

$= \int_0^{2\pi} \int_{r=0}^{a} \rho r^3 \, dr \, d\theta$

Inner: $\int_0^a \rho r^3 \, dr = \left[\rho \frac{r^4}{4}\right]_0^a = \rho \frac{a^4}{4}$

Outer: $\int_0^{2\pi} \rho \frac{a^4}{4} \, d\theta = \left[\frac{\rho}{4} \cdot a^4 \cdot \theta\right]_0^{2\pi} = \frac{2\pi\rho a^4}{4}$ or $\frac{\pi\rho a^4}{2}$

Note here that the total mass is

$\underbrace{\pi a^2}_{\text{area of circle}}(\rho) \Rightarrow M = \rho\pi a^2$

area of circle $\times$ density

and we see $I_0 = \rho\pi a^2 \cdot \frac{a^2}{2} = \frac{Ma^2}{2}$

Note here, by symmetry that $I_x = I_y$

Thus, since $I_0 = I_x + I_y$, we have $I_0 = 2I_x$

Thus $I_x = \frac{Ma^2}{4}$, same as I_y.

Application: Probability

Ex: If joint probability density function for X and Y is given by

$$f(x,y) = \begin{cases} C(x+2y) & \text{if } 0 \le x \le 10,\ 0 \le y \le 10 \\ 0 & \text{otherwise} \end{cases}$$

Then

(a) Find the constant value C

(b) Find the prob. that X is smaller than 7 while Y is bigger than 2.

ie compute $P(X \leq 7, Y \geq 2)$