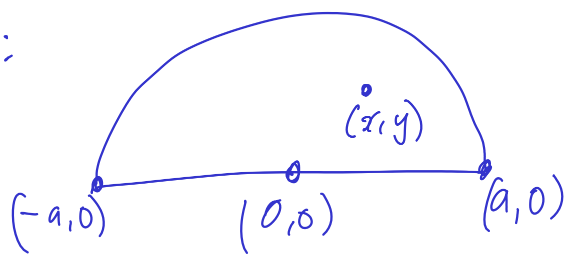
W5L2 : More Applications of Double Integrals

Ex (Stewart) A lamina has density $\rho(x,y)$ proportional to the distance of (x,y) from the origin.
This lamina is semi-circular of radius a.
Find its c.o.m.

Sol^n:



$$\rho(x,y) = K\sqrt{x^2+y^2}$$

where K is some constant of proportionality.

Note: shape of lamina and form of ρ both suggest using polar coords

D here is $\{(x,y)\in\mathbb{R}^2 : x = r\cos\theta,\ y = r\sin\theta;\ 0\le r\le a,\ 0\le\theta\le\pi\}$

and $\rho = Kr$

Recall c.o.m. is $(\bar{x},\bar{y})\in\mathbb{R}^2$,

$$\bar{x} = \frac{1}{m}\iint_D \rho x\, dA \ ; \quad \bar{y} = \frac{1}{m}\iint_D \rho y\, dA \ ; \quad m = \iint_D \rho\, dA$$

where $\rho = \rho(x,y)$.

Here $m = \iint_D Kr.\, r\,dr\,d\theta$ $\qquad \boxed{dA = r\,dr\,d\theta}$

$$= \int_{r=0}^{a}\int_{\theta=0}^{\pi} Kr^2\, d\theta\, dr$$

Inner: $\int_0^{\pi} Kr^2 d\theta = \left[Kr^2\theta\right]_0^{\pi} = Kr^2\pi - Kr^2(0) = Kr^2\pi$

Outer: $\displaystyle\int_0^a Kr^2\pi\, dr = \left[K\pi\frac{r^3}{3}\right]_0^a = \frac{K\pi a^3}{3} - 0$

Thus $\quad m = \dfrac{K\pi a^3}{3}$

Now, $\displaystyle\bar{x} = \frac{1}{m}\iint_D x\rho\, dA = \frac{1}{m}\iint_D \underbrace{r\cos\theta}_{x}.\underbrace{Kr}_{\rho}.\underbrace{r\,dr\,d\theta}_{dA}$

$$= \frac{1}{m}\int_0^a \int_{\theta=0}^{\pi} Kr^3\cos\theta\, d\theta\, dr$$

$=$ Inner: $\displaystyle\int_0^{\pi} Kr^3\cos\theta\, d\theta = \left[Kr^3\sin\theta\right]_{\theta=0}^{\pi}$

$$= Kr^3\sin\pi - Kr^3\sin 0$$
$$= 0 - 0$$

Thus $\quad \bar{x} = 0$

[We could have observed that $\bar{x} = 0$ by the symmetries in the problem.]

$$\bar{y} = \frac{1}{m}\iint_D y\rho\, dA = \frac{1}{m}\int_{r=0}^{a}\int_{\theta=0}^{\pi} r\sin\theta.\,Kr.\,r\,dr\,d\theta$$

$$= \frac{1}{m}\int_{r=0}^{a}\int_{\theta=0}^{\pi} Kr^3\sin\theta\, d\theta\, dr$$

Inner: $\displaystyle\int_0^{\pi} Kr^3\sin\theta\, d\theta = \left[-Kr^3.\cos\theta\right]_0^{\pi}$

$$= \left(-Kr^3(-1)\right) - \left(-Kr^3(1)\right)$$
$$= 2Kr^3$$

Outer: $\displaystyle\int_0^a 2Kr^3\, dr = \left[2K\frac{r^4}{4}\right]_0^a = \frac{2Ka^4}{4}$

So $\bar{y} = \frac{1}{m} \cdot \frac{Ka^4}{2} = \frac{3}{K\pi a^3} \cdot \frac{Ka^4}{2}$ $\left(m = \frac{K\pi a^3}{3} \text{ here} \right)$

$= \frac{3a}{2\pi}$

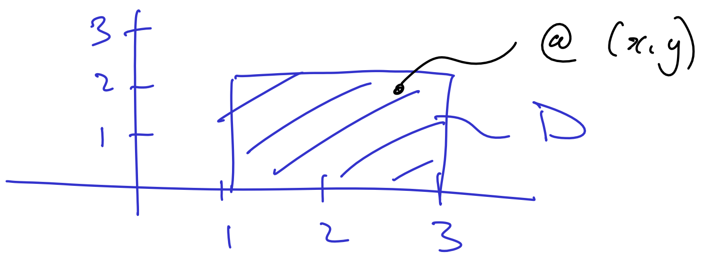
So $(\bar{x}, \bar{y}) = \left(0, \frac{3a}{2\pi}\right)$

<u>Ex</u> (Stewart) Electric charge is distributed over the rectangle $1 \le x \le 3$, $0 \le y \le 2$ so that the charge density σ (sigma) at (x,y) is

$$\sigma(x,y) = 2xy + y^2 \qquad (\text{Coulombs}/m^2)$$

<u>Task: Find the total charge Q</u>

<u>Soln</u>:



Total charge $Q = \iint_D \sigma(x,y)\, dA$ $\left\{ M = \iint_D \rho\, dA \right\}$

ie $Q = \int_{x=1}^{3} \int_{y=0}^{2} 2xy + y^2 \, dy\, dx = \ldots$

exercise.