Limits

If $f(x,y) \to L_1$
as $(x,y) \to (a,b)$ along path C_1

and $f(x,y) \to L_2$
as $(x,y) \to (a,b)$ along path C_2,

where $L_1 \neq L_2$

then: $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

Ex Show $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$

does not exist.

Solution

Lets first approach $(0,0)$
along the x-axis
i.e. $y=0$

Then

$$\lim_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{x^2-y^2}{x^2+y^2}$$

$$= \lim_{x\to 0} \frac{x^2}{x^2} = 1$$

Now approach $(0,0)$ along the y-axis

i.e. $x=0$

$$\lim_{\substack{(x,y)\to(0,0)\\ x=0}} \frac{x^2-y^2}{x^2+y^2}$$

$$= \lim_{y\to 0} \frac{-y^2}{y^2} = -1.$$

Since f has two different limits along two different lines,

the given limit does not exist.

Ex

If $f(x,y) = \dfrac{xy}{x^2+y^2}$,

does: $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exist?

Solution

Along x-axis $f(x,y) \to 0$

Along y-axis $f(x,y) \to 0$

Although we have obtained identical limits along the axes, that does not show that the limit is zero.

Let's now approach $(0,0)$ along another line, say $y=x$

$$\frac{xy}{x^2+y^2} \to \frac{x(x)}{x^2+x^2} = \frac{x^2}{2x^2} = \frac{1}{2}$$

Since we have obtained different limits along different paths the given limit does not exist

Ex If $f(x,y) = \frac{xy^2}{x^2+y^4}$

does $\lim_{(x,y)\to(0,0)} f(x,y)$ exist?

Solution

Along $y = mx$

$$\frac{xy^2}{x^2+y^4} = \frac{x(mx)^2}{x^2+(mx)^4} \rightarrow \frac{x\,m^2x^2}{x^2} = m^2x \to 0$$

but along

$x = y^2$

$$\frac{xy^2}{x^2+y^4} = \frac{(y^2)(y^2)}{(y^2)^2+y^4} = \frac{y^4}{y^4+y^4} = \frac{y^4}{2y^4} = \frac{1}{2}$$

So the limit does not exist.

Definition:

Let f be a function of two variables whose domain D includes points arbitrarily close to (a,b)

Then we say that the limit of $f(x,y)$ as (x,y) approaches (a,b) is L and we write

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

if

$\forall\ \epsilon > 0$

$\exists\ \delta$ such that

$$|(x,y)-(a,b)| < \delta$$

$$\Rightarrow \quad |f(x,y)-L| < \epsilon .$$

Ex Find $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$

Like the last examples we could show that

$$\frac{3x^2y}{x^2+y^2} \to 0$$

along any line through the origi[n]

This does not prove that the given limit is 0.

The limits along $y=x^2$ and $x=y^2$ also turn out to be 0

So we begin to suspect that the limit does exist and is equal to 0.

Let $\epsilon > 0$

We want to find $\delta > 0$ such that

$$0 < \sqrt{x^2+y^2} < \delta$$

$$\Rightarrow \left| \frac{3x^2y}{x^2+y^2} - 0 \right| < \epsilon$$

That is

$$0 < \sqrt{x^2+y^2} < \delta$$

$$\Rightarrow \frac{3x^2|y|}{x^2+y^2} < \epsilon$$

Notice

$x^2 \leq x^2+y^2$ as $y^2 \geqslant 0$

So $\frac{x^2}{x^2+y^2} \leq 1$

and therefore

$$\epsilon = \frac{3x^2|y|}{x^2+y^2} \leq 3|y| = 3\sqrt{y^2} \leqslant 3\sqrt{x^2+y^2}$$

Thus

if we choose $\delta = \frac{\epsilon}{3}$

and let $0 < \sqrt{x^2+y^2} < \delta$

Let us check

then $\left| \dfrac{3x^2y}{x^2+y^2} - 0 \right| \leq 3\sqrt{x^2+y^2}$

$$< 3(\delta) = 3\left(\frac{\epsilon}{3}\right) = \epsilon$$

CONTINUITY

Definition

A function of two variables is called <u>continuous</u> at (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b).$$

We say f is continuous on D if f is continuous at every point (a,b) in D.

Ex $f(x,y) = x^2y^3 + 3x + 4y^2$ is a polynomial so it is continuous everywhere.

Ex

Any rational function is continuous on its domain

$$g(x,y) = \frac{2xy+1}{x^2+y^2}$$

is continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$.

Ex

$$f(x,y) = \begin{cases} \dfrac{3x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

We know that $f(x,y)$ is continuous for $(x,y) \neq (0,0)$ as it is a rational function there.

Also from the last limit example we saw

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$$

As $\lim\limits_{(x,y)\to(0,0)} \dfrac{3x^2y}{x^2+y^2} = f(0,0)$

$f(x,y)$ is continuous at $(0,0)$ and so it is continuous on $\mathbb{R}$

Partial Derivatives

If f is a function of two variables

partial derivatives are the functions $f_x \left(\frac{\partial f}{\partial x}\right)$ and $f_y \left(\frac{\partial f}{\partial y}\right)$ defined by:

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Ex If $f(x,y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2,1)$ and $f_y(2,1)$

Solution

$$f_x(x,y) = 3x^2 + 2xy^3$$
$$f_x(2,1) = 3(2)^2 + 2(2)(1)^3 = 16$$

$$f_y(x,y) = 3x^2y^2 - 4y$$
$$f_y(2,1) = 3(2)^2(1)^2 - 4(1) = 8$$

Ex

If $f(x,y) = \sin\left(\frac{x}{1+y}\right)$

calculate $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$.

Solution

$$f_x(x,y) = \frac{\partial f}{\partial x} = \frac{1}{1+y}\cos\left(\frac{x}{1+y}\right)$$

$$f_y(x,y) = \frac{\partial f}{\partial y} = -\frac{x}{(1+y)^2}\cos\left(\frac{x}{1+y}\right)$$

Ex

Find f_x f_y and f_z if

$$f(x, y, z) = e^{xy} \ln(z)$$

Solution

$$f_x = y\, e^{xy} \ln(z)$$

$$f_y = x\, e^{xy} \ln(z)$$

$$f_z = \frac{e^{xy}}{z}$$

Second partial derivatives

If $z = f(x, y)$, we have the following notation

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$(f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

Ex

Find the second partial derivatives of

$$f(x,y) = x^3 + x^2 y^3 - 2y^2$$

Solution

$$f_x(x,y) = 3x^2 + 2xy^3$$

$$f_y(x,y) = 3x^2y^2 - 4y$$

$$f_{xx}(x,y) = 6x + 2y^3$$

$$f_{xy}(x,y) = 6xy^2$$

$$f_{yx}(x,y) = 6xy^2$$

$$f_{yy}(x,y) = 6x^2y - 4$$

Clairaut's Theorem

Suppose f is defined on a disc D that contains the point (a,b). If the function f_{xy} and f_{yx} are both continuous on D,

Then $f_{xy}(a,b) = f_{yx}(a,b)$

Chain rule

Suppose that u is a differentiable function of n variables $x_1, x_2, \dots, x_n$ and each x_j is a function of the m variables $t_1, t_2, \dots, t_m$.

Then u is a function of $t_1, t_2, \dots, t_m$

and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \dots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}.$$

Ex

If $u(x, y, z) = x^4 y + y^2 z^3$,

where $x = rse^t$, $y = rs^2e^{-t}$

and $z = r^2 s \sin(t)$,

find $\frac{\partial u}{\partial s}$ when $r = 2$, $s = 1$, $t = 0$

Solution

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s}$$

$$\Rightarrow \frac{\partial u}{\partial s} = (4x^3y)(re^t) + (x^4 + 2yz^3)(2rse^{-t}) + (3y^2z^2)(r^2\sin(t))$$

$= 192$ when $r = 2$, $s = 1$ and $t = 0$

Gradient vector

Definition:

The gradient vector of $f(x, y)$ at a point $P_0(x_0, y_0)$ is the vector

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$$

obtained by evaluating the partial derivatives of f at P_0

For $f(x, y, z)$ at $P_0(x_0, y_0, z_0)$

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

Definition:

The tangent plane to $f(x, y, z) = c$ at $P_0(x_0, y_0, z_0)$ is

$$0 = f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0)$$

Ex

Find the tangent plane to the surface

$$f(x, y, z) = x^2 + y^2 + z - 9$$

at the point $P_0(1, 2, 4)$

Solution

$$(\vec{\nabla} f)_{P_0} = (2x\,\hat{i} + 2y\,\hat{j} + \hat{k})_{P_0}$$

$$= 2\hat{i} + 4\hat{j} + \hat{k}$$

The tangent plane is:

$$2(x-1)+4(x-2)+(z-4)=0$$

Ex

Find the plane tangent to the surface

$$z = x\cos y - y e^{x} \quad \text{at } (0,0,0)$$

Solution

To find an equation for the plane tangent to a smooth surface $z = f(x,y)$ at a point $P_0(x_0, y_0, z_0)$ where $z = f(x,y)$, we first observe that the equation $z = f(x,y)$ is equivalent to $f(x,y) - z = 0$

The surface $z = f(x,y)$ is therefore the zero level surface of the function

$$F(x,y,z) = f(x,y) - z$$
$$= x\cos y - y e^{x} - z$$

$$\Rightarrow \left(\frac{\partial F}{\partial x}\right)_{P_0} = \left(\cos y - y e^{x}\right)_{P_0} = 1$$

$$\left(\frac{\partial F}{\partial y}\right)_{P_0} = -x\sin y - e^{x} = -1$$

$$\left(\frac{\partial F}{\partial z}\right)_{P_0} = -1.$$

$\Rightarrow$ The tangent plane is therefore:

$$(1)(x-0) + (-1)(y-0) + (-1)(z-0) = 0$$

$$x - y - z = 0$$

The Directional Derivative

If $f(x,y)$ is differentiable in an open region containing $P_0(x_0, y_0)$, then

$$\left(D_{\vec{u}} f\right)_{P_0} = \left(\vec{\nabla} f\right)_{P_0} \cdot \hat{u}$$

Ex

Find the derivative of

$$f(x,y) = x e^{y} + \cos(xy)$$

at the point $(2,0)$ in the direction of $\vec{v} = 3\hat{i} - 4\hat{j}$

Solution

$$\vec{v} = 3\hat{i} - 4\hat{j}$$

$$|\vec{v}| = \sqrt{(3)^2 + (-4)^2} = \sqrt{9+16} = \sqrt{25} = 5$$

$$\hat{v} = \frac{3}{5}\hat{i} - \frac{4}{5}\hat{j}$$

$$\left(f_x\right)_{P_0} = \left(e^{y} - y\sin(xy)\right)_{(2,0)} = e^{0} - 0 =$$

$$(f_y)_{P_0} = \left(xe^{y} - x\sin(xy)\right)_{(2,0)} = 2$$

$$\Rightarrow \left(D_{\vec{v}} f\right)_{P_0} = (i + 2j) \cdot \left(\frac{3}{5}\hat{i} - \frac{4}{5}\hat{j}\right)$$

$$= -1.$$

Properties of the Directional Derivative:

1. The function f increases most rapidly when

$\cos\theta = 0$

or when $\vec{u}$ is in the direction of $\vec{\nabla f}$.

$$D_{\vec{u}} f = |\vec{\nabla f}| \cos(0) = |\vec{\nabla f}|$$

2. f decreases most rapidly in the direction of $-\vec{\nabla f}$

3.

Any direction $\vec{u}$ orthoganal to a gradient $\vec{\nabla f} \neq \vec{0}$

is a direction of zero change in f.

<u>Ex</u>

Find the direction in which

$$f(x,y) = \frac{x^2}{2} + \frac{y^2}{2}$$

(a) Increases most rapidly at the point (1,1)

(b) Decreases most rapidly at the point (1,1)

(c) What are the directions of zero change at (1,1).

<u>Solution</u>

(a)
$$\vec{\nabla f} = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$$
$$= x\hat{i} + y\hat{j}$$
$$(\vec{\nabla f})_{(1,1)} = \hat{i} + \hat{j}$$

Need $\vec{u}$ to take the same direction as $(\vec{\nabla} f)_{(1,1)}$

i.e. $\hat{\imath} + \hat{\jmath}$

$$|\vec{u}| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

so $\hat{u} = \frac{1}{\sqrt{2}}(\hat{\imath} + \hat{\jmath})$

(b) The function decreases most rapidly in the direction of $-\vec{\nabla} f$ at (1,1 which is

$$-\frac{1}{\sqrt{2}}\hat{\imath} - \frac{1}{\sqrt{2}}\hat{\jmath}$$

(c) The directions of zero change at (1, 1) are the directions orthogonal to $\vec{\nabla} f$:

$$\hat{n} = -\frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath}$$

$$-\hat{n} = \frac{1}{\sqrt{2}}\hat{\imath} - \frac{1}{\sqrt{2}}\hat{\jmath}.$$

Exercise

(a) Find the derivative of

$f(x,y,z) = x^3 - xy^2 - z$

at $P_0(1,1,0)$

in the direction of $\vec{v} = 2\hat{i} - 3\hat{j} + 6$

(b) In what directions does f change most rapidly at P_0, and what are the rates of change in these directions.

Theorem:

- If $f(x,y)$ has a local MAXIMUM or MINIMUM value
- at an interior point (a,b) of its domain
- and if the first partial derivatives exist there,

then $f_x(a,b) = 0$ and $f_y(a,b) = 0$

Definition

An interior point of the domain of a function $f(x,y)$

- where both f_x and f_y are zero
- or where one or both of f_x and f_y do not exist
- is a critical point

Definition

A differentiable function $f(x,y)$ has a saddle point at a critical point (a,b)

- if in every open disc centred at (a,b)
- there are domain points (x,y) where $f(x,y) > f(a,b)$
- and domain points (x,y) where $f(x,y) < f(a,b)$.

The corresponding point $(a, b, f(a,b))$ on the surface $z = f(x,y)$ is called a saddle point of the surface.

Ex

Find the critical points of

$$f(x,y) = x^2 + y^2 - 2x - 6y + 14$$

Solution

$$f_x = 2x - 2$$

$$f_y = 2y - 6$$

if $f_x = 0 \Rightarrow 2x - 2 = 0$

$2x = 2 \Rightarrow x = 1$

if $f_y = 0 \Rightarrow 2y - 6 = 0$

$2y = 6 \Rightarrow y = 3$

$\therefore (1,3)$ is the only critical point

Second derivative test.

Suppose the second partial derivatives of f are continuous on a disc with centre (a,b), and suppose that

$$f_x(a,b) = 0 \quad \text{and} \quad f_y(a,b) = 0.$$

Let

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - (f_{xy})^2$$

with all second derivatives evaluated at (a,b)

(a) If $D > 0$ and $f_{xx}(a,b) > 0$ then $f(a,b)$ is a <u>local MINIMUM</u>

(b) If $D > 0$ and $f_{xx}(a,b) < 0$ then $f(a,b)$ is a <u>local MAXIMUM</u>

(c) If $D < 0$ then $f(a,b)$ is not a local MAXIMUM or MINIMUM — it is a saddle point.

Ex

Find the local MAXIMUM and MINIMUM values and saddle points of

$$f(x,y) = x^4 + y^4 - 4xy + 1$$

$$f_x = 4x^3 - 4y$$

$$f_y = 4y^3 - 4x$$

$f_x = 0 \iff 4x^3 - 4y = 0$

$\Rightarrow x^3 - y = 0 \Rightarrow x^3 = y$ (A)

$f_y = 0 \iff 4y^3 - 4x = 0$

$\Rightarrow y^3 - x = 0 \Rightarrow y^3 = x$ (B)

Substituting (A) into (B)

$$(x^3)^3 = x$$

$$x^9 - x = 0$$

$$x(x^8 - 1) = 0$$

$$x(x^4 - 1)(x^4 + 1) = 0$$

$$x(x^2 - 1)(x^2 + 1)(x^4 + 1) = 0$$

$$x(x-1)(x+1)(x^2+1)(x^4+1)=0$$

$$x=-1,\ 0,\ 1$$

So f_x and f_y are both zero at

using (A) i.e. $y=x^3$

$(-1,-1)$, $(0,0)$, $(1,1)$.

$f_{xx}=12x^2$ $\quad f_{xy}=-4$ $\quad f_{yy}=12y^2$

$$D=(12x^2)(12y^2)-(-4)^2$$
$$=144x^2y^2-16$$

$D(0,0)=-16<0$ $\therefore$ Saddle point

$D(1,1)=128>0$, $f_{xx}(1,1)=12>0$

$\therefore$ $(1,1)$ is a local MINIMUM

$D(-1,-1)=128>0$, $f_{xx}(-1,-1)=12>0$

$\therefore$ $(-1,-1)$ is a local MINIMUM

Ex

Find the shortest distance from the point $(1, 0, -2)$ to the plane $x+2y+z=4$

SOLUTION

$$d = \sqrt{(x-1)^2+y^2+(z+2)^2}$$

$$\Rightarrow d^2 = (x-1)^2+y^2+(z+2)^2$$

$$z = 4 - x - 2y$$

$$f(x,y) = d^2 = (x-1)^2 + y^2 + (6-x-2y)^2$$

$$f_x = 2(x-1) + 2(6-x-2y)(-1) = 0$$

$$\Rightarrow 2x-2 - 12 + 2x + 4y = 0$$

$$4x + 4y = 14$$

$$\text{or} \quad 2x+2y = 7 \quad \text{—— (A)}$$

$$f_y = 2y + 2(6-x-2y)(-2)$$

$$\Rightarrow 2y - 24 + 4x + 8y = 0$$

$$4x + 10y = 24$$

$$2x + 5y = 12 \quad \text{— (B)}$$

$(B)-(A) \Rightarrow \quad 3y = 5$

$y = 5/3$

$(A) \quad 2x + 2(5/3) = 7$

$6x + 10 = 21$

$6x = 11$

$x = 11/6$

$z = 4 - x - 2y$

$z = 4 - \frac{11}{6} - 2(5/3)$

$\frac{24 - 11 - 20}{6} = -\frac{7}{6}$

$\left(\frac{11}{6}, \frac{5}{3}, -\frac{7}{6}\right)$

$f_{xx} = 4 \quad f_{xy} = 4 \quad f_{yy} = 10$

$\left.\begin{array}{l} D = (4)(10) - 4^2 = 24 > 0 \\ f_{xx} = 4 > 0 \end{array}\right\} \therefore$ MINIMUM at $\left(\frac{11}{6}, \frac{5}{3}, -\frac{7}{6}\right)$

$d = \sqrt{\left(\frac{11}{6}-1\right)^2 + \left(\frac{5}{3}\right)^2 + \left(-\frac{7}{6}+2\right)^2}$

$= \sqrt{\left(\frac{5}{6}\right)^2 + \left(\frac{5}{3}\right)^2 + \left(\frac{5}{6}\right)^2} = \frac{5}{6}\sqrt{1+2^2+1}$

$= \frac{5}{6}\sqrt{6} = \frac{5\sqrt{6}}{6}$