Equation of the line in $\mathbb{R}^3$

First, in $\mathbb{R}^2$ to obtain the equation of a line, we need

a point on the line + the slope

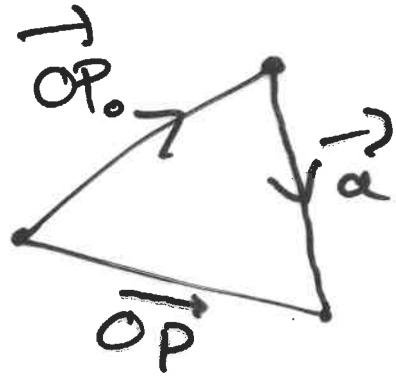
Likewise in $\mathbb{R}^3$ we need

a point on the line + "a slope".

We will find the equation of a line in $\mathbb{R}^3$ that contains the point $\vec{r}_0$ and is parallel to the vector $\vec{v}$.

- Let $\vec{r}_0$ have vector representation $\overrightarrow{OP_0}$
- Let $\vec{r}$ be an arbitrary point on the line

with vector representation $\overrightarrow{OP}$



By the triangular law

$$\overrightarrow{OP_0} + \vec{a} = \overrightarrow{OP}$$

or $\quad \vec{a} = \overrightarrow{OP} - \overrightarrow{OP_0} \quad *$

$\overrightarrow{OP} - \overrightarrow{OP_0}$ lies along the line (is parallel to line)

From $*$ $\vec{a}$ is parallel to line

So $\vec{a} = t\vec{v}$

so $*$ becomes $t\vec{v} = \vec{r} - \vec{r_0}$

so the vector equation is $\boxed{\vec{r} = \vec{r_0} + t\vec{v}}$ $\quad -\infty < t < \infty$

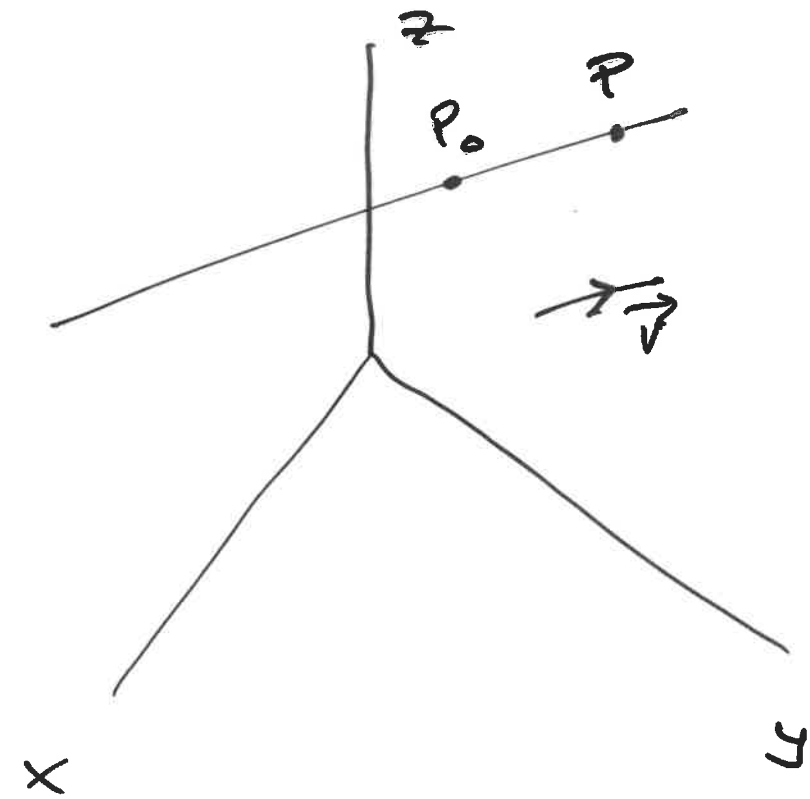
Looking at components, if

$$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \vec{r_0} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Parametric equation for the line

So
$$\begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases}$$

or $$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \quad (= t)$$



<u>Ex</u>

Find the parametric equation for the line through

$(-2, 0, 4)$

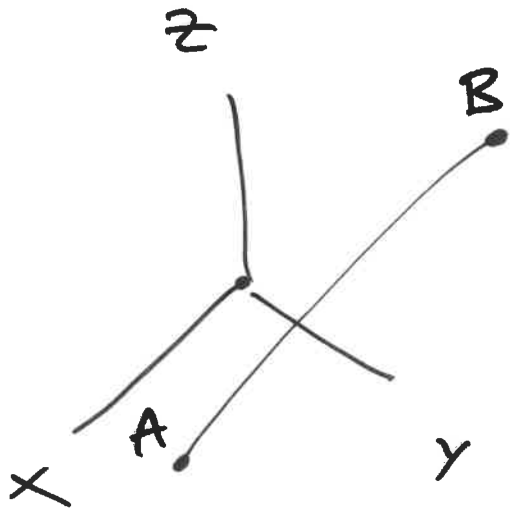
parallel to $(2, 4, -2)$

Solution

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 4 \end{pmatrix} + t\begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix}$$

or
$$\begin{cases} x = -2 + 2t \\ y = 4t \\ z = 4 - 2t \end{cases}$$

Ex Find the parametric equation for the line through $P(-3, 2, -3)$ and $Q(1, -1, 4)$



$\vec{AB}$ is parallel to line

So $\vec{PQ}$ is parallel to line

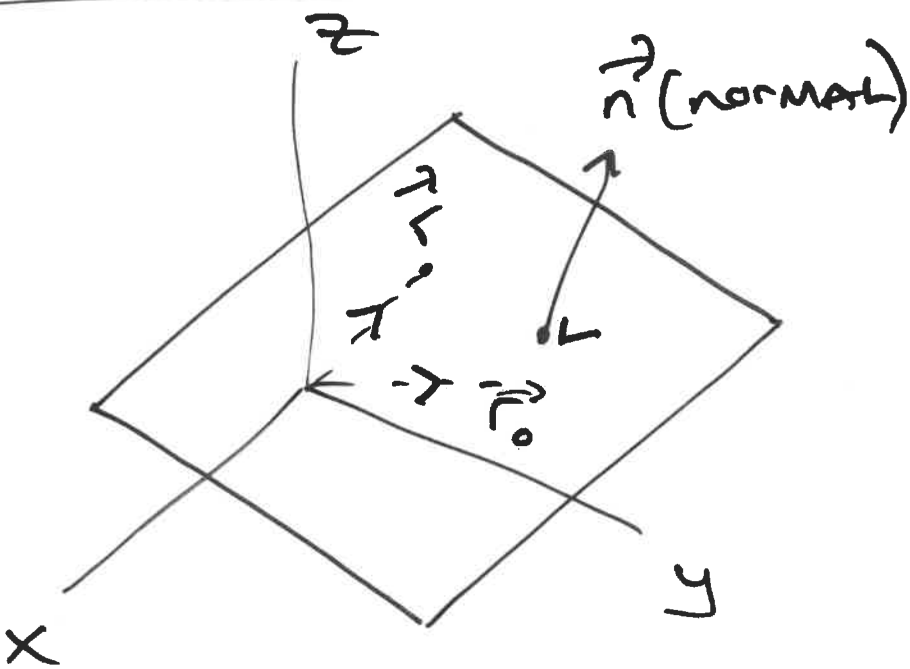
$$\vec{PQ} = \vec{Q} - \vec{P}$$
$$= \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} - \begin{pmatrix} -3 \\ 2 \\ -3 \end{pmatrix}$$

$$= \begin{pmatrix} 1+3 \\ -1-2 \\ 4+3 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ 7 \end{pmatrix}$$

So the equation of the line is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \\ -3 \end{pmatrix} + t \begin{pmatrix} 4 \\ -3 \\ 7 \end{pmatrix}$$

The equation of a plane.



- $\vec{r}$ is an arbitrary point on the plane
- $\vec{r_0}$ is a known point on the plane
- $\vec{n}$ is a normal vector to the plane

So $\overrightarrow{r_0 r}$ is perpendicular to $\vec{n}$

as $\overrightarrow{r_0 r}$ lies along the plane

So $\overrightarrow{r_0 r} \bullet \vec{n} = 0$

or $$\boxed{(\vec{r} - \vec{r_0}) \bullet \vec{n} = 0}$$

e.g. if $\vec{r} = (x, y, z)$

$\vec{r_0} = (x_0, y_0, z_0)$

$\vec{n} = (a, b, c)$

then

$$\begin{pmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{pmatrix} \bullet \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

or $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

or $ax + by + cz = \underbrace{ax_0 + by_0 + cz_0}_{\text{Known constant}}$

or $$\boxed{ax + by + cz = d.}$$

This is a linear equation in x, y and z.

Ex Find the equation of the plane through the point $(2, 4, -1)$ with normal $(2, 3, 4)$.

$(r - r_0) \cdot \vec{n} = 0$

$$\Rightarrow \left[\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}\right] \cdot \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = 0$$

$$\Rightarrow 2(x-2) + 3(y-4) + 4(z+1) = 0$$

$$\Rightarrow 2x - 4 + 3y - 12 + 4z + 4 = 0$$

$$\Rightarrow 2x + 3y + 4z = 12$$

Exercise

Find the equation of the plane through the points:

$P(1, 3, 2)$
$Q(3, -1, 6)$
$R(5, 2, 0)$

and verify your answer (by substitution).

A vector-valued function (or vector function) is a function whose domain is a set of real numbers and whose range is a set of vectors

Definition

The limit of a vector function $\vec{r}$ is defined by taking the limit of its components.

If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

then $\lim_{t \to a} \vec{r}(t) =$

$$\left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\rangle$$

Ex Find $\lim_{t \to 0} \vec{r}(t)$, where

$$\vec{r}(t) = (1+t^3)\,\hat{i} + te^{-t}\,\hat{j} + \frac{\sin t}{t}\,\hat{k}$$

$$\lim_{t \to 0} \vec{r}(t) =$$

$$\left\langle \lim_{t \to 0} (1+t^3),\ \lim_{t \to 0} te^{-t},\ \lim_{t \to 0} \frac{\sin t}{t} \right\rangle$$

$$= \langle 1, 0, 1 \rangle$$

Definition

A vector function $\vec{r}$ is continuous at a if

$$\lim_{t \to a} \vec{r}(t) = \vec{r}(a).$$

The derivative $\vec{r}\,'(t)$ of a vector function $\vec{r}(t)$ is defined in the same way as for real valued functions i.e.

$$\frac{d\vec{r}}{dt} = \vec{r}\,'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h)-\vec{r}(t)}{h}$$

The vector

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}(t)|} \qquad \left(= \hat{T}(t)\right)$$

is called the <u>unit tangent vector</u>

Theorem :

If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

$= f(t)\,\hat{\imath} + g(t)\,\hat{\jmath} + h(t)\,\hat{k}$

where f, g and h are differentiable functions, then

$$\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle$$

$$= f'(t)\,\hat{\imath} + g'(t)\,\hat{\jmath} + h'(t)\,\hat{k}.$$

Ex For $\vec{r}(t) = \langle 1+t^3, te^{-t}, \sin 2t \rangle$

(a) Find the derivative of $\vec{r}(t)$

(b) Find the unit tangent vector at the point $t=0$.

Solution

(a) Note 1 $\frac{d(1+t^3)}{dt} = 3t^2$

Note 2 $\frac{d(te^{-t})}{dt} = t\frac{d(e^{-t})}{dt} + e^{-t}\frac{d(t)}{dt}$

$= -te^{-t} + e^{-t} = e^{-t}(1-t)$

$\left(\frac{d(uv)}{dt} = u\frac{dv}{dt} + v\frac{du}{dt} \right)$

Note 3 $\frac{d(\sin 2t)}{dt} = \frac{d(\sin 2t)}{d(2t)} \frac{d(2t)}{dt}$

$= \cos(2t) \cdot 2$

$= 2\cos(2t)$

So $\vec{r}\,'(t) = \langle 3t^2, e^{-t}(1-t), 2\cos(2t) \rangle$

(b) $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$

$$\vec{T}(0) = \frac{\vec{r}\,'(0)}{|\vec{r}\,'(0)|} = \frac{\langle 0, 1, 2 \rangle}{\sqrt{0^2+1^2+2^2}}$$

$$= \frac{1}{\sqrt{5}} \langle 0, 1, 2 \rangle$$

Differentiation rules:

Suppose $\vec{u}$ and $\vec{v}$ are differentiable vector functions,

- c is a scalar
- and f is a real valued function

then

1. $\dfrac{d}{dt}\left[\vec{u}(t) + \vec{v}(t)\right] = \vec{u}(t) + \vec{v}(t)$.

2. $\dfrac{d}{dt}\left[f(t)\,\vec{u}(t)\right] = f'(t)\,\vec{u}(t) + f(t)\,\vec{u}\,'(t)$

3. $\dfrac{d}{dt}\left[c\,\vec{u}(t)\right] = c\,\vec{u}\,'(t)$

4 $\frac{d}{dt}\left[\vec{u}(t) \cdot \vec{v}(t)\right]$

$= \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t)$

5 $\frac{d}{dt}\left[\vec{u}(t) \times \vec{v}(t)\right]$

$= \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)$

6 $\frac{d}{dt}\left[\vec{u}(f(t))\right] = f'(t)\, \vec{u}'(f(t))$

Chain rule.

<u>Ex</u> Show:

If a curve lies on a circle with centre the origin, then the tangent vector is always perpendicular to the position vector.

Solution:

$$\vec{r}(t) \cdot \vec{r}(t) = |\vec{r}(t)|^2 = c^2 \quad *$$

(c is the radius)

Using 4 above

$$\frac{d}{dt}\left[\vec{r}(t) \cdot \vec{r}(t)\right]$$

$$= \vec{r}\,'(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{r}\,'(t)$$

$$= 2\vec{r}(t) \cdot \vec{r}\,'(t)$$

i.e. $\frac{d}{dt}\left[\vec{r}(t) \cdot \vec{r}(t)\right] = 2\vec{r}(t)\vec{r}\,'(t)$

From $*$ $\frac{d}{dt}\left[c^2\right] = 2\vec{r}(t) \cdot \vec{r}\,'(t)$

c^2 is a constant
so its derivative is zero

So $0 = \vec{r}(t) \cdot \vec{r}\,'(t)$

i.e. the position vector is perpendicular to the tangent vector.

Exercise Show that

$$\vec{r}(t) = \sin(t)\,\hat{i} + \cos(t)\,\hat{j} + \sqrt{3}\,\hat{k}$$

has constant length and is orthogonal to its derivative.

Definition:

The **integral** of $\vec{r}(t)$

If $\vec{r}(t) = f(t)\,\hat{i} + g(t)\,\hat{j} + h(t)\,\hat{k}$, then definite integral

$$\int_a^b \vec{r}(t)\,dt = \int_a^b f(t)\,dt\;\hat{i} + \int_a^b g(t)\,dt\;\hat{j} + \int_a^b h(t)\,dt\;\hat{k}.$$

the indefinite integral

$$\int \vec{r}(t)\,dt = \int f(t)\,dt\;\hat{i} + \int g(t)\,dt\;\hat{j} + \int h(t)\,dt\;\hat{k}$$

<u>Exercise</u> Find the definite integral

$$\int_0^{\pi/2} \vec{r}(t)\,dt \quad \text{with}$$

$$\vec{r}(t) = 2\cos t\,\hat{i} + \sin t\,\hat{j} + 2t\,\hat{k}$$

Exercise Find the indefinite integral

$$\int e^t\,\hat{i} + 2t\,\hat{j} + (\ln t)\,\hat{k}\,dt.$$

Integration by parts (review)

<u>Ex</u> Evaluate $\int x^2 e^x\,dx$

<u>Ans</u>: use $\int u\,dv = uv - \int v\,du$

with $u = x^2$ $dv = e^x dx$

So $du = 2x dx$ $v = e^x$

So $\int x^2 e^x dx = (x^2)(e^x) - \int e^x 2x^1 dx$ *

The new integral is easier to deal with as the power is 1 smaller.

To evaluate $\int x^1 e^x dx$

we use integration by parts

for a 2nd time

i.e. use $\int u dv = uv - \int v du$

with $u = x$ $dv = e^x dx$

So $du = dx$ $v = e^x$

So $\int x e^x dx = (xe^x) - \int e^x dx$

$= xe^x - e^x$

Substituting this into *

$$\int x^2 e^x dx = x^2 e^x - 2[xe^x - e^x]$$

$$= x^2 e^x - 2e^x(x-1) + C_1$$

Exercise, Find

(a) $\int e^x \cos(x) dx$

(b) $\int \ln(x)$