

1.4.2 : Integration by parts

In this section we discuss the technique of **integration by parts**, which is essentially a reversal of the product rule of differentiation.

Example 22

Find $\int x \cos x \, dx$.

Solution How could $x \cos x$ arise as a derivative?

Well, $\cos x$ is the derivative of $\sin x$. So, if you were differentiating $x \sin x$, you would get $x \cos x$ but according to the product rule you would also get another term, namely $\sin x$.

Conclusion: $\int x \cos x \, dx = x \sin x + \cos x + C$.

Managing this process

What happened in this example was basically that the product rule was reversed. This process can be managed in general as follows. Recall from differential calculus that if u and v are expressions involving x , then

$$(uv)' = u'v + uv'.$$

Suppose we integrate both sides here with respect to x . We obtain

$$\int (uv)' \, dx = \int u'v \, dx + \int uv' \, dx \implies uv = \int u'v \, dx + \int uv' \, dx.$$

This can be rearranged to give the **Integration by Parts Formula** :

$$\int uv' \, dx = uv - \int u'v \, dx.$$

Here is the first example again, handled according to this scheme.

Example 23

Use the integration by parts technique to determine $\int x \cos x \, dx$.

Solution: Write

$$\begin{aligned} u &= x & v' &= \cos x \\ u' &= 1 & v &= \sin x \end{aligned}$$

Then

$$\begin{aligned} \int x \cos x \, dx &= \int uv' \, dx = uv - \int u'v \, dx \\ &= x \sin x - \int 1 \sin x \, dx \\ &= x \sin x + \cos x + C. \end{aligned}$$

An antiderivative of $\ln x$

Example 24

Determine $\int \ln x \, dx$.

Solution: Let $u = \ln x$, $v' = 1$.

Then $u' = \frac{1}{x}$, $v = x$.

$$\begin{aligned} \int \ln x \, dx &= \int uv' \, dx = uv - \int u'v \, dx \\ &= x \ln x - \int \frac{1}{x} x \, dx \\ &= x \ln x - x + C. \end{aligned}$$

Note: This example shows that sometimes problems which are not obvious candidates for integration by parts can be attacked using this technique.

Two Rounds of Integration by Parts

Sometimes two applications of the integration by parts formula are needed, as in the following example.

Example 25

Evaluate $\int x^2 e^x \, dx$.

Solution: Let $u = x^2$, $v' = e^x$. Then $u' = 2x$, $v = e^x$.

$$\begin{aligned} \int x^2 e^x \, dx &= \int uv' \, dx = uv - \int u'v \, dx \\ &= x^2 e^x - \int 2x e^x \, dx \\ &= x^2 e^x - 2 \int x e^x \, dx. \end{aligned}$$

Let $I = \int x e^x \, dx$.

Two rounds (continued)

Let $I = \int x e^x \, dx$

To evaluate I apply the integration by parts formula a second time.

$$\begin{aligned} u &= x & v' &= e^x \\ u' &= 1 & v &= e^x. \end{aligned}$$

Then $I = \int x e^x \, dx = x e^x - \int e^x \, dx = x e^x - e^x + C$. Finally

$$\int x^2 e^x \, dx = x^2 e^x - 2x e^x + 2e^x + C.$$

An Example of Another Type

The next example shows another mechanism by which a second application of the integration by parts formula can succeed where the first is not enough.

Example 26

Determine $\int e^x \cos x \, dx$.

Solution Let

$$\begin{aligned} u &= e^x & v' &= \cos x \\ u' &= e^x & v &= \sin x. \end{aligned}$$

Then

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx.$$

$\int e^x \cos x \, dx$ (continued)

For $\int e^x \sin x \, dx$: Let

$$\begin{aligned} u &= e^x & v' &= \sin x \\ u' &= e^x & v &= -\cos x. \end{aligned}$$

Then

$$\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx,$$

and

$$\begin{aligned} \int e^x \cos x \, dx &= e^x \sin x - \left(-e^x \cos x + \int e^x \cos x \, dx \right) \\ \Rightarrow 2 \int e^x \cos x \, dx &= e^x \sin x + e^x \cos x + C \\ \Rightarrow \int e^x \cos x \, dx &= \frac{1}{2}(e^x \sin x + e^x \cos x) + C \end{aligned}$$

A Definite Integral

Example 27

Evaluate $\int_0^1 (x+3)e^{2x} \, dx$.

Solution: Write $u = x+3$, $v' = e^{2x}$; $u' = 1$, $v = \frac{1}{2}e^{2x}$

$$\begin{aligned} \int_0^1 (x+3)e^{2x} \, dx &= \int uv' \, dx = (uv)|_0^1 - \int_0^1 u'v \, dx \\ &= \left. \frac{x+3}{2}e^{2x} \right|_0^1 - \frac{1}{2} \int_0^1 e^{2x} \, dx \\ &= \left. \frac{x+3}{2}e^{2x} \right|_0^1 - \frac{1}{2} \times \left. \frac{1}{2}e^{2x} \right|_0^1 \\ &= \frac{4}{2}e^2 - \frac{3}{2}e^0 - \frac{1}{4}e^2 + \frac{1}{4}e^0 = \frac{7}{4}e^2 - \frac{5}{4}. \end{aligned}$$