

## 1.4.1 Substitution - Reversing the Chain Rule

The Chain Rule of Differentiation tells us that in order to differentiate the expression  $\sin x^2$ , we should regard this expression as  $\sin(\text{"something"})$  whose derivative (with respect to "something") is  $\cos(\text{"something"})$ , then multiply this expression by the derivative of the "something" with respect to  $x$ . Thus

$$\frac{d}{dx}(\sin x^2) = \cos x^2 \frac{d}{dx}(x^2) = 2x \cos x^2.$$

Equivalently

$$\int 2x \cos x^2 dx = \sin x^2 + C.$$

In this section, through a series of examples, we consider how one might go about reversing the differentiation process to get from  $2x \cos x^2$  back to  $\sin x^2$ .

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## How Substitution Works

### Example 17

Determine  $\int 2x\sqrt{x^2+1} dx$ .

**Solution** Notice that the integrand involves both the expressions  $x^2+1$  and  $2x$ . Note also that  $2x$  is the derivative of  $x^2+1$ .

1 Introduce the notation  $u$  and set  $u = x^2+1$ .

2 Note  $\frac{du}{dx} = 2x$ ; rewrite this as  $du = 2x dx$ .

3 Then

$$\int 2x\sqrt{x^2+1} dx = \int \sqrt{x^2+1}(2x dx) = \int u^{\frac{1}{2}} du = \frac{2}{3}u^{\frac{3}{2}} + C.$$

4 So

$$\int 2x\sqrt{x^2+1} dx = \frac{2}{3}(x^2+1)^{\frac{3}{2}} + C.$$

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## Substitution and definite integrals

### Example 18

Determine  $\int_0^\pi \cos^3 x \sin x dx$  (from 2015 Summer paper)

**Solution:** Write  $u = \cos x$ . Then

$$\frac{du}{dx} = -\sin x, \quad du = -\sin x dx, \quad \sin x dx = -du.$$

Change variables:  $\int_0^\pi \cos^3 x \sin x dx = -\int_{x=0}^{x=\pi} u^3 du$ . Limits of integration: When  $x=0$ ,  $u = \cos x = \cos 0 = 1$ . When  $x=\pi$ ,  $u = \cos x = \cos \pi = -1$ . Our integral becomes:

$$-\int_{u=1}^{u=-1} u^3 du = \frac{u^4}{4} \Big|_{u=-1}^{u=1} = \frac{1}{4} - \frac{(-1)^4}{4} = 0.$$

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## Substitution and Definite Integrals - more examples

### Example 19

Evaluate  $\int_0^1 \frac{5r}{(4+r^2)^2} dr$ .

**Solution** To find an antiderivative, let  $u = 4+r^2$ .

Then  $\frac{du}{dr} = 2r$ ,  $du = 2r dr$ ;  $5r dr = \frac{5}{2} du$ .

So

$$\int \frac{5r}{(4+r^2)^2} dr = \frac{5}{2} \int \frac{1}{u^2} du = \frac{5}{2} \int u^{-2} du.$$

Thus

$$\int \frac{5r}{(4+r^2)^2} dr = -\frac{5}{2} \times \frac{1}{u} + C,$$

and we need to evaluate  $-\frac{5}{2} \times \frac{1}{u}$  at  $r=0$  and at  $r=1$ . We have two choices.

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## Two Choices

1 Write  $u = 4+r^2$  to obtain

$$\begin{aligned} \int_0^1 \frac{5r}{(4+r^2)^2} dr &= -\frac{5}{2} \times \frac{1}{4+r^2} \Big|_{r=0}^{r=1} \\ &= -\frac{5}{2} \times \frac{1}{4+1^2} - \left( -\frac{5}{2} \times \frac{1}{4+0^2} \right) \\ &= -\frac{5}{2} \times \frac{1}{5} + \frac{5}{2} \times \frac{1}{4} \\ &= \frac{1}{8}. \end{aligned}$$

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## ... Alternatively

2. Alternatively, write the antiderivative as  $-\frac{5}{2} \times \frac{1}{u}$  and replace the limits of integration with the corresponding values of  $u$ .

When  $r=0$  we have  $u = 4+0^2 = 4$ .

When  $r=1$  we have  $u = 4+1^2 = 5$ .

Thus

$$\begin{aligned} \int_0^1 \frac{5r}{(4+r^2)^2} dr &= -\frac{5}{2} \times \frac{1}{u} \Big|_{u=4}^{u=5} \\ &= -\frac{5}{2} \times \frac{1}{5} - \left( -\frac{5}{2} \times \frac{1}{4} \right) \\ &= \frac{1}{8}. \end{aligned}$$

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## Example 20

Determine

$$\int_1^4 \frac{1}{x + \sqrt{x}} dx.$$

Solution: Write

$$\int_1^4 \frac{1}{x + \sqrt{x}} dx = \int_1^4 \frac{1}{\sqrt{x}(\sqrt{x} + 1)} dx.$$

Now write  $u = \sqrt{x} + 1$ . Then  $\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2} \frac{1}{\sqrt{x}} \Rightarrow \frac{1}{\sqrt{x}} dx = 2du$ .

Then

$$\begin{aligned} \int_1^4 \frac{1}{\sqrt{x}(\sqrt{x} + 1)} dx &= \int_{x=1}^{x=4} \frac{2}{u} du = \int_{u=2}^{u=3} \frac{2}{u} du = 2 \ln u \Big|_2^3 \\ &= 2(\ln 3 - \ln 2) = 2 \ln \frac{3}{2}. \end{aligned}$$

## Example

Determine

$$\int_1^4 \frac{1}{x + \sqrt{x}} dx.$$

**Note on the exam:** This question was not answered well. It was not intended to be particularly difficult or tricky. Only about five people in the whole class answered it correctly. Many candidates made very fundamental and serious errors in algebra before attempting the integration, for example rewriting  $\frac{1}{x + \sqrt{x}}$  as  $\frac{1}{x} + \frac{1}{\sqrt{x}}$ . No credit could be awarded in such a case since the error rendered the rest of the question meaningless. So BE CAREFUL.

## More Examples

## Example 21

Determine  $\int (1 - \cos t)^2 \sin t dt$ 

**Question:** How do we know what expression to extract and refer to as  $u$ ?

Really what we are doing in this process is changing the integration problem in the variable  $t$  to a (hopefully easier) integration problem in a new variable  $u$  - there is a change of variables taking place.

There is no easy answer but with practice we can develop a sense of what might work. In this example the integrand involves the expression  $1 - \cos t$  and also its derivative  $\sin t$ . This is what makes the substitution  $u = 1 - \cos t$  effective for this problem.

**NOTE:** There are more examples of the substitution technique in the lecture notes.