

# MA133C & MA160

## Calculus 1

### Lecture 9



## Rate of change

The White Rabbit is late for his non-birthday party, and is running to get there. To exit his rabbit hole he has to go through a tunnel 500m long. He reaches the end of the tunnel at minute 3 of his journey. What was the rabbit's **average velocity** in the tunnel?

Let's write  $r(t)$  for the distance run by the rabbit at time  $t$  (in seconds), so  $r(0) = 0$  and  $r(180) = 500$ . The average velocity through the tunnel is (in m/s):

$$\frac{r(180) - r(0)}{180 - 0} = \frac{500}{180} \approx 2.78$$

## Rate of change

At the end of the tunnel he finds his bicycle and decides to cycle to the party. With his bike he can travel in  $t$  seconds a distance of  $b(t) = t^2/4$  meters, starting from  $t = 0$ . He meets the Cheshire Cat after 2 minutes and arrives at his destination in 5 minutes.

- What was his average velocity between seeing the Cat and arriving at destination?

We can mimic what we did in the case of the tunnel, using this time the given (quadratic) function and the relevant times:

$$b(120) = \frac{14400}{4} = 3600, \quad b(300) = \frac{90000}{4} = 22500$$

and therefore in those three minutes

$$\text{average velocity} = \frac{b(300) - b(120)}{300 - 120} = \frac{22500 - 3600}{180} = \frac{18900}{180} = 105\text{m/s}$$

## Average velocity and average rate of change

We can write down the general formula for the average velocity of a motion with position at time  $t$  given by  $s(t)$ :

### Average velocity

$$\text{Average velocity between } t = t_1 \text{ and } t = t_2 = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

We shall see that a similar formula can be used to evaluate, in general, the **average rate of change** of some function with respect to some interval.

## Instant velocity and limits of functions

Going back to the White Rabbit cycling to his party...

- ▶ What was his **instant velocity** when he passed the Cat?

We can interpret the instant velocity  $v$  as the **limit** of the average velocity taken over smaller and smaller intervals of time:

$$\begin{aligned}v(120) &= \lim_{h \rightarrow 0} \frac{b(120 + h) - b(120)}{120 + h - 120} = \lim_{h \rightarrow 0} \frac{1}{4} \frac{(120 + h)^2 - 120^2}{h} = \\ &= \frac{1}{4} \lim_{h \rightarrow 0} \frac{14400 + 240h + h^2 - 14400}{h} = \frac{1}{4} \lim_{h \rightarrow 0} \frac{h(240 + h)}{h} = \frac{1}{4} \cdot 240 = 60\text{m/s}\end{aligned}$$

Note that the instant velocity is a function of time.

In fact, it is the **derivative** of the position as a function of time.

## Instant velocity and instant rate of change

In the same spirit as before, we can write down the general formula for the instant velocity of a motion with position at time  $t$  given by  $s(t)$ :

### Instant velocity at time $t$

$$s'(t) = \lim_{h \rightarrow 0} \frac{s(t+h) - s(t)}{h}$$

This formula can be used to evaluate, in general, the **instant rate of change** of a function at some point. In other words, to evaluate the **derivative** of the function at some point.

## Derivative of a function: definition and examples

We are ready to give the formal definition of the derivative of a function at a point:

### Derivative at a point

The derivative of a function  $f$  at a point  $x = a$ , denoted by  $f'(a)$ , is

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.

**Example (1)** Evaluate the derivative of  $f(x) = 2x^2 + 3$  at  $x = 1$ .

$$\begin{aligned} f'(1) &= \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \rightarrow 0} \frac{2(1+h)^2 + 3 - (2 \cdot 1^2 + 3)}{h} \\ &= \lim_{h \rightarrow 0} \frac{2h^2 + 4h + 2 + 3 - 2 - 3}{h} = \lim_{h \rightarrow 0} \frac{2h(h+2)}{h} = 4 \end{aligned}$$

## Derivative of a function: definition and examples

**Example (2)** Evaluate the derivative of  $g(x) = \frac{2}{x}$  at  $x = 3$ .

**Example (3)** Evaluate the derivative of  $u(x) = \frac{1-x}{2+x}$  at  $x = 0$ .

**Tomorrow:** geometric interpretation and first practical rules.