MA133C & MA160
Calculus 1

Lecture 7

Recap, 1

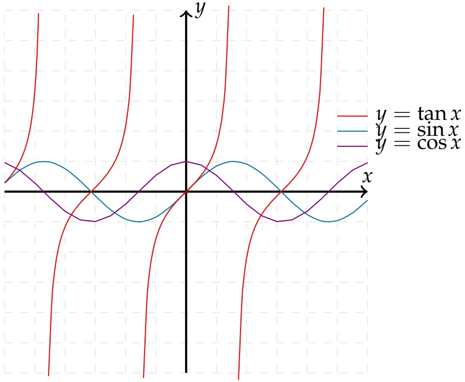
- **Composition of functions:** is an operation defining a new function $f \circ g$ from two given functions f, g. Its domain and graph will depend on the given functions.
- **Absolute value:** a piecewise defined function

$$f(x) = |x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0. \end{cases}$$

Example. Given $f(x) = x^2 - 1$ and $g = |3x|$, compute $f \circ f, g \circ f, f \circ g$. What is the range of each of them?

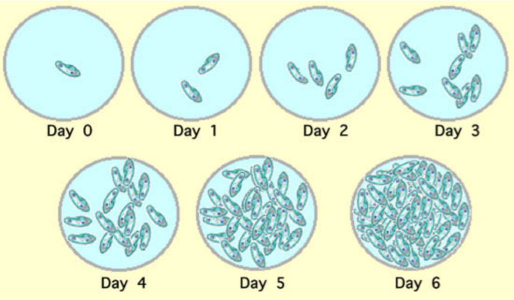
Recap, 2

- **Trigonometric functions:** cosine, sine, tangent. Cosine and sine are periodic functions (with period 2π) defined on $\mathbb{R}$. The tangent function is the ratio of sine and cosine. It is defined on $\mathbb{R} \setminus \{x : x = k + \pi/2, k \in \mathbb{Z}\}$.



Fast-growing populations

Suppose in an experiment we observe the following behaviour:



and we want to write down a function for the number of bacteria B on day n:

$$B(0) = 1,\ B(1) = 2,\ B(2) = 4,\ B(3) = 8,\ B(4) = 16\ldots$$

The function modelling this "very fast growth" is the **exponential function**.

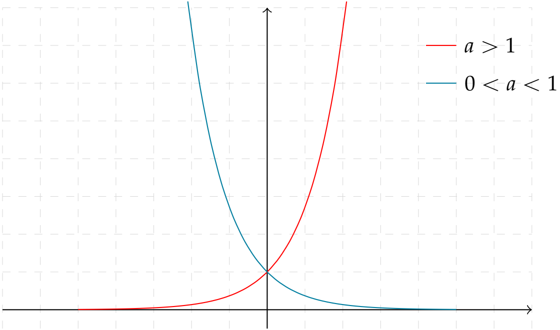
The exponential function: definition and graph

Let $a > 0$ be a real number.

Definition: exponential function

The exponential function with base a is the function $f(x) = a^x$. It is defined for all real values of x.

This is how it looks like for $a > 1$ and for $0 < a < 1$:



The exponential function: special values and basic properties

Now the real question is: what does this mean?

As we mentioned earlier, this is a transcendental function: we don't expect to be able to "compute" its values in general.

However, for certain values of the variable x, we can actually calculate the value of $f(x) = a^x$ algebraically:

- if n is a positive integer then $\boxed{a^n = a \cdot a \cdots a}$ (with n factors) and $\boxed{a^{-n} = 1/a^n}$
- if $x = 0$ then $\boxed{a^x = a^0 = 1}$
- if $x = p/q$ is a rational number (with $q > 0$) then $\boxed{a^x = (\sqrt[q]{a})^p}$

Moreover, the following law of exponents holds for all real numbers x, z:

$$\boxed{a^{x+z} = a^x \cdot a^z.}$$

⚠ Do not confuse exponential functions (where the variable is the exponent and the base is fixed) with power functions!

More properties and a special base

- If $0 < a < 1$ the exponential function with base a is a **decreasing function**, a function for which $f(x_1) > f(x_2)$ if $x_1 < x_2$. Moreover, its limit at $+\infty$ is zero:

$$\lim_{x \to +\infty} a^x = 0 \qquad \text{and} \quad y = 0 \text{ is a right horizontal asymptote.}$$

- If $a > 1$ the exponential function with base a is an **increasing function**, a function for which $f(x_1) < f(x_2)$ if $x_1 < x_2$. Moreover, its limit at $-\infty$ is zero:

$$\lim_{x \to -\infty} a^x = 0 \qquad \text{and} \quad y = 0 \text{ is a left horizontal asymptote.}$$

A special role is played by one particular choice of base: the number e. This is a real number, its value (correct to the third decimal) is $e \approx 2.718\ldots$.

We will call the function $f(x) = e^x$ simply "the exponential function".

Applications: population growth

When $a > 1$ the function $f(x) = a^x$ grows at a very high pace. It is therefore used in models of population growth[1]. Going back to our bacteria...

Example. Under ideal conditions a certain bacteria polulation is known to double every three hours. Suppose that there are initially 100 bacteria.

1. What is the size of the population after 15 hours?
2. What is the size of the population after t hours?
3. Estimate the size of the population after 20 hours.
4. Graph the population function and estimate the time for the population to reach $50,000$.

[1] We will learn more advanced methods to deduce these models as an application of differentiation.

Exercises

Exercise 1. Let $f(x) = e^{x+3} - 1$. Determine domain, range, x- and y-axis intercepts of f and sketch its graph.

Exercise 2. Let $g(x) = \frac{1}{2}e^{-x} + 2$. Determine domain, range, x- and y-axis intercepts of g and sketch its graph.