MA133C & MA160
Calculus 1

Lecture 5

Recap

Last week:

- **Algebraic functions:** the "most general" type of functions that can be defined only using algebraic operations (addition, multiplication, taking reciprocal and taking roots).

  Examples. $f(x) = \dfrac{\sqrt{x^2 - 2 - 3x}}{x}, \qquad g(x) = \sqrt[5]{x+1} + x^5 - 12$

  Their domain will depend on the domains of the various functions appearing in their expressions. Their graphs can be very different.

- **Limits at infinity** (for now: only of polynomials and rational functions): if a function is defined on some large domain of the form $(a, +\infty)$ we sometimes want to answer the question: "What's the behaviour of the function as the variable becomes very large?". The tool we use to answer this question is the limit at infinity. On the next slides we collect some practical rules to compute some limits at infinity.

Limits at infinity of rational functions 1

Given a rational function

$$f(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0},$$

then whether it has or not a limit at infinity will depend on the **degrees** of the polynomials $p(x)$ and $q(x)$.

1. $\boxed{n < m}$ This means that the denominator grows faster than the numerator as x goes to ∞. In this case

   $$\lim_{x\to\infty} f(x) = 0 \qquad \text{and the function has a horizontal asymptote } y = 0.$$

Example

$$\lim_{x\to+\infty} \frac{x-1}{2x^2+9x-1} = 0.$$

Limits at infinity of rational functions 2

Given a rational function

$$f(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0},$$

then whether it has or not a limit at infinity will depend on the **degrees** of the polynomials $p(x)$ and $q(x)$.

2 $\boxed{n = m}$ This means that the numerator and denominator behave roughly in the same way as x goes to ∞. In this case the limit at infinity will be the ratio of the leading coefficients:

$$\lim_{x\to\infty} f(x) = \frac{a_n}{b_m}$$ and the function has a horizontal asymptote $y = a_n/b_m$.

Example

$$\lim_{x\to+\infty} \frac{1-3x+9x^2}{1-3x^2} = \frac{9}{-3} = -3$$

Limits at infinity of rational functions 3

Given a rational function

$$f(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0},$$

then whether it has or not a limit at infinity will depend on the **degrees** of the polynomials $p(x)$ and $q(x)$.

3 $\boxed{n > m}$ In this case the numerator grows faster than the denominator and the function has no finite limit at infinity.

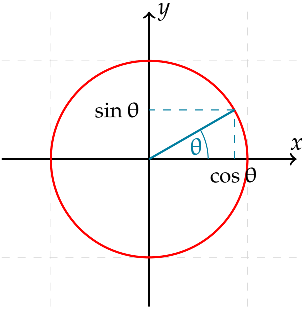
Example

$$\lim_{x \to +\infty} \frac{2x^2 + 9x - 1}{x - 1} = +\infty.$$

⚠ In all cases, when calculating the limit make sure your rational function is given by a single fraction. Also, when extracting the leading coefficients remember to include the sign!

Back to our catalog: trigonometric functions

So far we have dealt with functions which can be defined via algebraic operations. Functions which cannot be defined in this way are called **transcendental**. These include trigonometric, inverse trigonometric, exponential, and logarithmic functions, and many more. We will start with the trigonometric functions.



The unit circle

The unit circle is the circle with equation

$$x^2 + y^2 = 1.$$

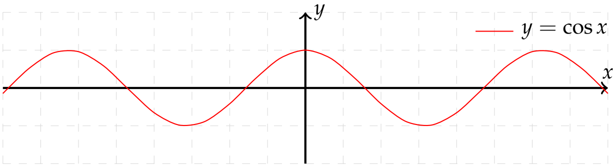
Given a real number θ define $\cos\theta$ and $\sin\theta$ to be the x-coordinate and y-coordinate, respectively, of the point on the unit circle that subtends an angle of radian measure θ with the positive x-axis.

Trigonometric functions: cosine

The **cosine** function is defined on the whole real line. Its range is $[-1, 1]$. This function is **periodic** of period 2π. This means that the identity

$$\cos(x) = \cos(x + 2k\pi) \text{ holds for all integer numbers } k.$$

Also, cosine is an even function, so $\cos(x) = \cos(-x)$ holds for all $x \in \mathbb{R}$.



Here are the values of this function for some useful angles (in radians):

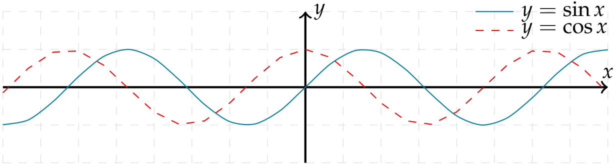
x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	π
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	-1

Trigonometric functions: sine

The **sine** function is defined on the whole real line. Its range is $[-1, 1]$. Like the cosine, sine is **periodic** of period 2π. This means that the identity

$$\sin x = \sin(x + 2k\pi) \text{ holds for all integer numbers } k.$$

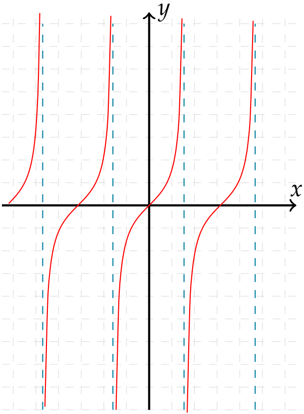
The sine function is an odd function, so $\sin x = -\sin(-x)$ holds for all $x \in \mathbb{R}$.



Here are the values of this function for some useful angles (in radians):

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	π
$\sin(x)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	0

Trigonometric functions: tangent



The **tangent function** is defined as:

$$\tan x = \frac{\sin x}{\cos x}.$$

As we can see from the graph and from its expression, the tangent function is not defined for all values of x for which $\cos x = 0$, namely the domain of the tangent function is $\mathbb{R} \setminus \{x : x = k + \pi/2, k \in \mathbb{Z}\}$

The tangent function is **periodic** of period π (the period is “shorter” than that of cos or sin!).