MA133C & MA160
Calculus 1

Lecture 4

Sketching graphs: new functions from old functions

When a function is obtained from a “basic” function by a simple transformation, the graph of the new function can also be sketched from the old one simply translating and/or stretching and/or reflecting it.

Here is a summary of simple transformations that can be applied to functions and the resulting transformations on the graphs. Suppose you have a function f and know how the graph of $y = f(x)$ looks like.

For $c > 0$:

On the **function**	On the **graph**
$y = f(x) + c$	shift the graph of $y = f(x)$ of “c units” upward
$y = f(x) - c$	shift the graph of $y = f(x)$ of “c units” downward
$y = f(x - c)$	shift the graph of $y = f(x)$ of “c units” to the right
$y = f(x + c)$	shift the graph of $y = f(x)$ of “c units” to the left
$y = -f(x)$	reflect the graph of $y = f(x)$ about the x-axis
$y = f(-x)$	reflect the graph of $y = f(x)$ about the y-axis

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For $c > 1$:

On the **function**	On the **graph**
$y = cf(x)$	stretch the graph of $y = f(x)$ vertically by a factor of c
$y = (1/c)f(x)$	shrink the graph of $y = f(x)$ vertically by a factor of c
$y = f(cx)$	shrink the graph of $y = f(x)$ horizontally by a factor of c
$y = f(x/c)$	stretch the graph of $y = f(x)$ horizontally by a factor of c

Example

$$h(x) = \frac{x+1}{x+2} = \frac{(x+2)-1}{x+2} = 1 - \frac{1}{x+2}$$

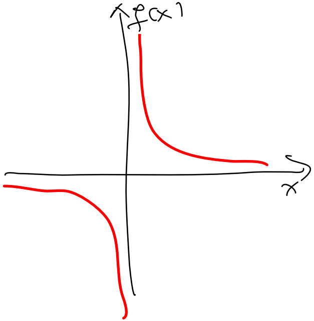
This tells us: we have to:

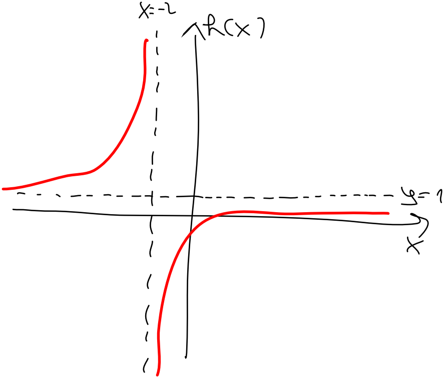
→ shift the graph of 2 to the left

→ reflect it about the x-axis

→ shift it of 1 upward.

If we call $f(x) = 1/x$ the reciprocal function, then $h(x) = 1 - f(x+2)$.





Limits: an introduction

Given a function, we are sometimes interested in its behaviour near points in which it is (possibly) not defined, or for very big or very small values of x. To this aim, we need to introduce the concept of **limit**.

Example 1. A tank contains 5000L of pure water. Brine that contains 30g of salt per liter of water is pumped into the tank at a rate of 25 L/min.

(1) Write down a function for the concentration $C(t)$ of salt after t minutes (in grams per liter).

(2) Sketch the graph of $C(t)$. (Hint: this is obtained by stretching and shifting a hyperbola.) For the model to make sense, what should the domain of C be?

(3) What happens to the concentration as t goes to ∞?

To answer the last question we will need to introduce the concept of **limit** at infinity.

Limits: definition

Limit at infinity

Suppose f is defined on some interval (a, ∞). We write

$$\lim_{x \to \infty} f(x) = L \qquad \text{“the } \textbf{limit of } f(x)\text{, as } x \text{ goes to } \infty\text{, equals } L\text{”}$$

if the values of f can be made arbitrarily close to L by taking x sufficiently large.
Similarly, if f is defined on some interval $(-\infty, a)$. We write

$$\lim_{x \to -\infty} f(x) = L \qquad \text{“the } \textbf{limit of } f(x)\text{, as } x \text{ goes to } -\infty\text{, equals } L\text{”}$$

if the values of f can be made arbitrarily close to L by taking x sufficiently large negative.

Example 1. The function giving the concentration of salt at time t in the previous example is:

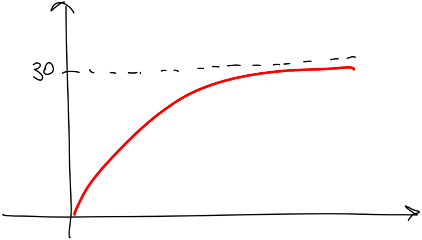
$$C(t) = \frac{30t}{200 + t}.$$

Does this function have a limit at $+\infty$?

Limits: horizontal asymptotes

$$C(t) = \lim_{t\to\infty} \frac{30t}{200+t} = \lim_{t\to\infty} \frac{t\cdot 30}{t\cdot(200/t+1)} = \lim_{t\to\infty} \frac{30}{1+200/t} = 30$$

If we sketch the graph of the function $y = C(t)$, we note that for large t it is very close to the horizontal line $y = 30$.



Horizontal asymptote

The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = L.$$

Examples

as x goes to ∞ $\frac{1}{3x}$ and $\frac{1}{2x}$ both become very small: they "tend to 0"

Evaluate the following limits:

- $\lim\limits_{x\to\infty} \frac{1-3x}{2x+1}$ $= \lim\limits_{x\to\infty} \frac{-3\cancel{x}(1-\frac{1}{3x})}{2\cancel{x}(1+1/2x)} \overset{\downarrow}{=} \lim\limits_{x\to\infty} \frac{-3(1-\frac{1}{3x})}{2(1+\frac{1}{2x})} = -\frac{3}{2}$
- $\lim\limits_{x\to\infty} \frac{x^2+x-2}{3-2x^2}$
- $\lim\limits_{x\to-\infty} \frac{x(x+2)^2}{x^3-1}$ $= \lim\limits_{x\to-\infty} \frac{x(x^2+4x+4)}{x^3-1} = \lim\limits_{x\to-\infty} \frac{\cancel{x^3}\left(1+\overset{\to 0}{\boxed{\frac{4x^2}{x^3}}}+\overset{\to 0}{\boxed{\frac{4x}{x^3}}}\right)}{\cancel{x^3}\left(1-\underset{\to 0}{\boxed{\frac{1}{x^3}}}\right)} = 1$
- $\lim\limits_{x\to-\infty} \frac{(x+2)^2}{x^3-1}$

we will collect a number of rules to help us evaluate limits at ∞ of rational functions.