

MA133C & MA160
Calculus 1

Lecture 19

Limits and derivatives: indeterminate forms

We have seen that sometimes the various laws of limits don't suffice to compute a limit. For instance, consider

$$\lim_{x \rightarrow -1} \frac{x^2 - 2x - 3}{x^3 + x^2}.$$

If we try and substitute the value $x = -1$ in the fraction we get "0/0". In a situation like this the limit may or may not exist. We call it an **indeterminate form of type $\frac{0}{0}$** .

In our example can use an *ad hoc* factorisation to get

$$\lim_{x \rightarrow -1} \frac{x^2 - 2x - 3}{x^3 + x^2} = \lim_{x \rightarrow -1} \frac{(x - 3)(x + 1)}{x^2(x + 1)} = \lim_{x \rightarrow -1} \frac{(x - 3)}{x^2} = \frac{-1 - 3}{1} = -4.$$

However, especially when we are trying to evaluate limits of transcendental functions, such methods do not work.

Limits and derivatives: indeterminate forms

Another type of indeterminate form arises when we evaluate limits at infinity (to decide whether the function admits a horizontal asymptote or not, for instance). An example is:

$$\lim_{x \rightarrow +\infty} \frac{e^x}{x^2}.$$

The rule for the limit of a quotient doesn't help, as both e^x and x^2 go to $+\infty$ as x goes to $+\infty$.

We have again an indeterminate form, which we call **indeterminate form of type $\frac{\infty}{\infty}$** .

Good news: there is a nice application of differentiation that gives us a rule to solve limits of this form: the **l'Hopital's Rule**.

Limits and derivatives: L'Hopital's Rule

L'Hopital's Rule

1. Suppose f and g are differentiable around a point a , that $g'(x) \neq 0$ near a (except possibly at a) and that at a we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Then:

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}.$$

2. Suppose f and g are differentiable when x is large (or large negative) and that at infinity we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Then:

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \frac{f'(x)}{g'(x)}.$$

Limits and derivatives: L'Hopital's Rule

L'Hopital's Rule

Informally, this means that the limit of a quotient of two functions is the same as the limit of the quotient of their derivatives, **provided that the given conditions are satisfied**.

Let's apply the rule to our two examples from before.

Example. (1) Compute

$$\lim_{x \rightarrow -1} \frac{f(x)}{g(x)} = \frac{x^2 - 2x - 3}{x^3 + x^2}.$$

First, let's verify that the l'Hopital's Rule is applicable. This is the limit of the quotient of two polynomials f and g (both differentiable). Moreover, the denominator has derivative $g'(x) = 3x^2 + 2x$ which is nonzero around -1 . Therefore we can apply our rule and get

$$\lim_{x \rightarrow -1} \frac{f(x)}{g(x)} = \lim_{x \rightarrow -1} \frac{f'(x)}{g'(x)} = \lim_{x \rightarrow -1} \frac{2x - 2}{3x^2 + 2x} = \frac{-4}{1} = -4.$$

L'Hopital's Rule: examples

Example. (2) Compute $\lim_{x \rightarrow +\infty} \frac{e^x}{x^2}$.

Here we are looking at an indeterminate form of type ∞/∞ . The function is a quotient of an exponential (differentiable) and a polynomial (also differentiable). Applying the rule we get

$$\lim_{x \rightarrow +\infty} \frac{e^x}{x^2} = \lim_{x \rightarrow +\infty} \frac{\frac{d}{dx}(e^x)}{\frac{d}{dx}(x^2)} = \lim_{x \rightarrow +\infty} \frac{e^x}{2x} = \frac{\infty}{\infty} \dots?$$

One application of the rule gives us again an indeterminate form. The good news here is that the right hand side above also satisfies the conditions for the rule, and therefore we can use it again:

$$\dots = \lim_{x \rightarrow +\infty} \frac{e^x}{2x} = \frac{\frac{d}{dx}(e^x)}{\frac{d}{dx}(2x)} = \lim_{x \rightarrow +\infty} \frac{e^x}{2} = \infty$$

The second application of the rule solved our limit, as ∞ divided by a constant is no indeterminate form.

L'Hopital's Rule: caution is needed



Compute

$$\lim_{x \rightarrow 0} \frac{\sin x}{1 + \cos x}.$$

Suppose we forget to check the conditions carefully, and as $\sin 0 = 0$ we decide to apply l'Hopital's Rule for this limit. What we get is

$$\lim_{x \rightarrow 0} \frac{\cos x}{\sin x} = \frac{1}{0}$$

This is **not** the result of our original limit. In fact, the denominator there is $1 + \cos x$ which is 2 when $x = 0$. So we didn't have an indeterminate form in the first place:

$$\lim_{x \rightarrow 0} \frac{\sin x}{1 + \cos x} = \frac{0}{2} = 0,$$

simply by substitution.

Sketching graphs: summary

The theory we developed about first and second derivatives, extreme point, limits helps us sketching an accurate graph of a function. Let's put together an algorithm to do this:

1. Determine the domain of f . This tells us *where* the function (and therefore our graph) lives.
2. Find the x - and y - intercepts, if they exist. This gives us some points through which the curve will pass.
3. (Symmetry, periodicity: it is not strictly needed, but very helpful to detect any of these in order to sketch the graph)
4. Asymptotes: if it makes sense (i.e. the domain of the function is unbounded to the right or to the left) compute

$$\lim_{x \rightarrow +\infty} f(x) \quad \text{and/or} \quad \lim_{x \rightarrow -\infty} f(x).$$

If one or both are finite limits, say L , then $y = L$ is a horizontal asymptote for the function.

Sketching graphs: summary

5. (Intervals of increase or decrease: these correspond, respectively, to intervals in which the first derivative is positive or negative.)
6. Local Maximum and Local Minimum Values: find all critical points and then study their nature using 5. (sign of first derivative) or using the second derivative test.
7. (Concavity: find the second derivative and study its sign to know where the function is concave upwards and where downwards. Knowing the sign of the second derivative at critical points where the first derivative is zero can be used for 6.)

Sketching graphs: examples

Following the steps in the summary, study the function

$$f(x) = \frac{2x^2}{x^2 - 1}.$$

1. Domain: all values of x such that $x^2 - 1 \neq 0$, that is $x^2 \neq 1$, so $x \neq 1$ and $x \neq -1$

$$\Rightarrow \text{Domain } (-\infty, -1) \cup (-1, 1) \cup (1, +\infty)$$

2. $f(0) = 0$ so $(0, 0)$ is the x - and y -intercept

$$3. \quad f(x) = \frac{2x^2}{x^2 - 1} \quad ; \quad f(-x) = \frac{2(-x)^2}{(-x)^2 - 1} = \frac{2x^2}{x^2 - 1} \quad * \text{ optional}$$

So f is an even function (its graph will be symmetric w.r.t. the y -axis)

4. horizontal asymptote: $\lim_{x \rightarrow \infty} \frac{2x^2}{x^2 - 1} = 2$ (limit of a rational function)

So $y = 2$ is a horizontal asymptote.

Sketching graphs: examples

$$\lim_{x \rightarrow 1^+} \frac{2x^2}{x^2-1} = +\infty, \quad \lim_{x \rightarrow 1^-} \frac{2x^2}{x^2-1} = -\infty; \quad \lim_{x \rightarrow -1^+} \frac{2x^2}{x^2-1} = -\infty, \quad \lim_{x \rightarrow -1^-} \frac{2x^2}{x^2-1} = +\infty$$

so $x=1$ and $x=-1$ are vertical asymptotes.

5. / 6. $f'(x) = \frac{4x(x^2-1) - 2x^2 \cdot 2x}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}$ which is 0 only at $x=0$

so $x=0$ is the only CP of f .

$$f''(x) = -4 \left[\frac{(x^2-1)^2 - x \cdot 2(x^2-1) \cdot 2x}{(x^2-1)^4} \right] = \frac{-4(x^2-1)(x^2-1-4x^2)}{(x^2-1)^4} = \frac{4(3x^2+1)}{(x^2-1)^3}$$

2nd derivative test for $x=0$:

$$f''(0) = \frac{4}{(-1)^3} = -4 < 0 \quad \text{so } x=0 \text{ is a local maximum}$$

this is a sketch of the graph of f

