

**MA133C & MA160**  
**Calculus 1**

Lecture 18

## Recap

- ▶ The maximum of a function  $f$  is the largest value it attains. The minimum is the smallest.
- ▶ Whether absolute extreme values exist or not depends on the function and the domain we consider.
- ▶ A continuous function attains extreme values in a closed interval  $[a, b]$ . Here's how to find them:
  1. Find the **critical** points of  $f$ : zeros of its first derivative and points at which the derivative does not exist.
  2. Find the values of  $f$  at the critical points of  $f$  in  $(a, b)$ .
  3. Find the values of  $f$  at the endpoints of the interval.
  4. Compare all the values: the largest of these is the absolute maximum value and the smallest is the absolute minimum value.
- ▶ Local extreme values are to be found among the critical points.
- ▶ If the function is nice enough then the second derivative test can help identifying local maxima and minima.
- ▶ The sign of the second derivative tells if the graph is concave upwards ( $f'' > 0$ ) or downwards ( $f'' < 0$ ).

## Example/Recap

$$\text{Let } f(x) = \frac{2x}{1+x+x^2}.$$

1. Find the domain of  $f$ . Decide whether it has any horizontal or vertical asymptotes.
2. Find and classify all local extreme values of the function  $f(x) = \frac{2x}{1+x+x^2}$ .
3. Determine the extreme values of  $f$  in the interval  $[0, 3]$

1. A rational function is defined for all values of  $x$  for which the denominator is nonzero. In our problem:

$$1+x+x^2=0 \Leftrightarrow x = \frac{-1 \pm \sqrt{1-4}}{2} \text{ which has no real solutions}$$

$\Rightarrow$  the domain of  $f$  is  $\mathbb{R}$ .

The graph of  $f$  has no vertical asymptotes (for a rational function they can occur at points where the denominator is 0).

Horizontal asymptotes: we have to evaluate  $\lim_{x \rightarrow \infty} f(x)$  and  $\lim_{x \rightarrow -\infty} f(x)$

## Example/Recap

These are both 0, as the numerator has degree lower than the degree of the denominator. So  $y=0$  is a horizontal asymptote for  $f$ .

2. local extreme values can only occur at critical points, which are pts where  $f'(x)=0$  or  $f'(x)$  does not exist.

$$\text{Here: } f'(x) = 2 \cdot \left[ \frac{1 \cdot (1+x+x^2) - x(2x+1)}{(1+x+x^2)^2} \right] = 2 \cdot \frac{1+x+x^2-2x^2-x}{(1+x+x^2)^2} = 2 \cdot \frac{(1-x^2)}{(1+x+x^2)^2}$$

which is 0 when  $x=1$  or  $x=-1$ .

We use the 2nd derivative test to decide whether  $f(1)$  and  $f(-1)$  are or not local extreme values, and if so to classify them.

$$f''(x) = 2 \cdot \frac{(-2x)(1+x+x^2)^2 - (1-x^2) \cdot 2 \cdot (1+x+x^2)(2x+1)}{(1+x+x^2)^4} = 2 \cdot \frac{\cancel{(1+x+x^2)}^2 \cdot (-2x) - (1-x^2) \cdot 2 \cdot (1+x+x^2)(2x+1)}{(1+x+x^2)^4} = 2 \cdot \frac{(-2x) - (1-x^2) \cdot 2 \cdot (2x+1)}{(1+x+x^2)^3} \cdot (2x^3 - 6x - 2)$$

$$= 4 \frac{(x^3 - 3x - 1)}{(1+x+x^2)^2}$$

the sign of  $f''$  at 1 and -1 tells us whether they are local min / local max / neither one.

$$f''(1) = \frac{4(1-3-1)}{3^2} = \frac{4 \cdot (-3)}{3^2} = -\frac{4}{3} < 0 \text{ so } f(1) \text{ is a local max}$$

$$f''(-1) = \frac{4(-1+3-1)}{(1-1+1)^2} = 4 > 0 \text{ so } f(-1) \text{ is a local min.}$$

3. over  $[0,3]$  we need to compare  $f(0), f(1), f(3)$ .

$f(0) = 0$  is the abs min  
 $f(1) = \frac{2}{3}$  is the abs max

# Mean Value Theorem

## Mean Value Theorem

If  $f$  is **nice enough** in the interval  $[a, b]$ <sup>1</sup> then there is a number  $c$  in  $(a, b)$  such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Remember the two main interpretations we gave of the first derivative of a function: **in terms of slope of the tangent line** and in terms of **instant velocity**. The mean value theorem has interpretations in both contexts.

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<sup>1</sup>that is: continuous in the closed interval  $[a, b]$ , differentiable in the open interval  $(a, b)$

## Mean Value Theorem: slopes

If the function is nice enough between  $a$  and  $b$ , then there is (at least) a point in the interval at which the slope of the tangent equals the slope of the secant through the endpoints.

### Example.

- ▶ Find the secant  $s$  to the graph of  $f(x) = x^3 + x - 1$  through its points with  $x = 0$  and  $x = 2$ .
- ▶ At which points (if any) in the interval  $[0, 2]$  is the tangent to the graph parallel to the line  $s$ ?
- ▶ Write an equation for the tangent at each of these points.

## Mean Value Theorem: slopes

Slope of secant through  $(a, f(a))$  and  $(b, f(b))$  is  $\frac{f(b) - f(a)}{b - a}$ . In our case:  $\frac{f(2) - f(0)}{2} = \frac{2^3 + 2 - 1 - (-1)}{2} = 5$

Our secant passes through  $(0, -1)$  and has slope 5, so its equation is  $y + 1 = 5(x - 0)$   
 $y = 5x - 1$

The Mean Value Thm says: there exists  $c \in [0, 2]$  such that

$$f'(c) = 5.$$

Let's find  $c$ :

$$f'(x) = 3x^2 + 1$$

so  $c$  is the solution of

$$3x^2 + 1 = 5 \quad \text{that is} \quad x^2 = \frac{4}{3}$$

$$\text{so} \quad x = \frac{2}{\sqrt{3}} \quad \text{or} \quad x = -\frac{2}{\sqrt{3}}$$

$$x = -\frac{2}{\sqrt{3}}$$

↗ This is not admissible because falls outside our interval  $[0, 2]$

$$\text{Now, } f\left(\frac{2}{\sqrt{3}}\right) = \frac{8}{3\sqrt{3}} + \frac{2}{\sqrt{3}} - 1 = \frac{14}{3\sqrt{3}} - 1$$

Our tangent passes through  $\left(\frac{2}{\sqrt{3}}, \frac{14}{3\sqrt{3}} - 1\right)$  and has slope 5:

$$y - \frac{14}{3\sqrt{3}} + 1 = 5\left(x - \frac{2}{\sqrt{3}}\right)$$

$$\text{so} \quad y = 5x - \frac{10}{\sqrt{3}} + \frac{14}{3\sqrt{3}} - 1$$

is the equation of our tangent.

## Mean Value Theorem: instant and average velocity

Here is a reformulation of the MVT in terms of instant and average velocity.

If the position in time between  $a$  and  $b$  is given by a nice enough function, then there is (at least) a point at which the instant velocity equals the average velocity between  $a$  and  $b$ .

**Example.** A particle is travelling along a straight line. Its position at time  $t$  is given by some differentiable function  $f(t)$ . If the position after 3 seconds is 2m and the velocity is always less than or equal than 2m/s, how far can the particle get after 10 seconds?

$$\begin{aligned} f(3) &= 2 && \text{How big can } f(10) \text{ be?} \\ \text{MVT says:} &&& \text{there is a point } c \text{ such that } f'(c) = \frac{f(10) - f(3)}{7} \\ \text{so } f'(c) \cdot 7 &= f(10) - 2 \end{aligned}$$

## Mean Value Theorem: instant and average velocity

$$\text{So } f(10) = f'(c) \cdot 7 + 2$$

If the velocity is always  $\leq 2$  then in particular it is so at  $t = c$ . Therefore

$$f(10) = f'(c) \cdot 7 + 2 \leq 7 \cdot 2 + 2 = 16$$

So 16m is the furthest point we can get at time 10sec.

## Exercise

$$\text{Let } f(x) = \frac{x^2 - 2x - 3}{x^3 + x^2}.$$

- ▶ Find and classify all critical points of  $f$ .
- ▶ Determine whether  $f$  has any horizontal or vertical asymptotes.

The critical points of  $f$  are  $\left\{ \begin{array}{l} x \text{ such that } f'(x) = 0 \\ x \text{ such that } f'(x) \text{ does not exist.} \end{array} \right.$

Here:

$$\begin{aligned} f'(x) &= \frac{(2x-2)(x^3+x^2) - (x^2-2x-3)(3x^2+2x)}{(x^3+x^2)^2} = \\ &= \frac{2x^4 + 2x^3 - 2x^3 - 2x^2 - 3x^4 - 2x^3 + 6x^3 + 4x^2 + 9x^2 + 6x}{x^4(x+1)^2} = \\ &= \frac{-x^4 + 4x^3 + 11x^2 + 6x}{x^4(x+1)^2} = \frac{-x^3 + 4x^2 + 11x + 6}{x^4(x+1)^2} \end{aligned}$$

## Exercise

$$= \frac{(6-x)(x+1)^2}{x^3(x+1)^2} = \frac{6-x}{x^3}$$

So  $f'(x)=0$  exactly when  $x=6$   
and  $f'(x)$  doesn't exist when  $x=0$

⚠ But  $x=0$  is not a critical point of  $f$ , as  $f$  itself is not defined for  $x=0$ !

So  $f$  admits only one critical point

$$f''(x) = \frac{-x^3 - 3x^2(6-x)}{x^6} = \frac{\cancel{x^2}(-x-18+3x)}{x^{\cancel{2}+4}} = \frac{2x-18}{x^4}$$

At  $x=6$  we have

$$f''(6) = \frac{12-18}{6^4} = -\frac{6}{6^4} = -\frac{1}{6^3} < 0 \quad \text{so } f(6) \text{ is a local max}$$