MA133C & MA160
Calculus 1

Lecture 17

Fermat's Theorem

While looking for local and global extreme values, we used the observation that if f is differentiable, then a local maximum (resp. minimum) is attained at a point with horizontal tangent. This is known as Fermat's Theorem:

Fermat's Theorem

If f has a local maximum or minimum at c and f is differentiable at c then $f'(c) = 0$.

⚠ This does not mean that all points at which the derivative is zero are either local maxima or local minima.

For instance, $f(x) = x^3$ has $f'(0) = 0$ but $f(0)$ is neither a local maximum nor a local minimum for f.

⚠ ⚠ Also, local maxima and minima can be attained at points where the function is not differentiable.

For instance, $f(x) = |x|$ has a local and absolute minimum 0 at $x = 0$ but it is not differentiable at $x = 0$.

Algorithm for extreme values in a closed interval

If f satisfies the hypotheses of the Extreme Value Theorem, we can find its extreme values following the simple steps:

Extreme values in a closed interval $[a, b]$: recipe

1. Find the **critical** points of f: zeros of its first derivative and points at which the derivative does not exist.
2. Find the values of f at the critical points of f in (a, b).
3. Find the values of f at the endpoints of the interval.
4. Compare all the values: the largest of these is the absolute maximum value and the smallest is the absolute minimum value.

Examples

1. Find the extreme values of $f(t) = te^{-t^2}$ in the interval $[-3, 3]$.

Following our algorithm:

1. Find zeros of 1st derivative: $f'(t) = (1-2t^2)e^{-t^2}$ which is 0 at $t_1 = -\frac{1}{\sqrt{2}}$ and $t_2 = \frac{1}{\sqrt{2}}$

   Are there any other critical pts? (i.e. pts where f is not differentiable...) No.

2. Evaluate f at the critical pts: $f(t_1) = -\frac{1}{\sqrt{2}}e^{-\frac{1}{2}}$; $f(t_2) = \frac{1}{\sqrt{2}}e^{-\frac{1}{2}}$

3. Evaluate f at the endpoints of the interval: $f(-3) = -3e^{-9}$; $f(3) = 3e^{-9}$

4. Compare all these values and choose the largest and the smallest.

$f(-3) \approx -0.00037$; $f(3) \approx 0.00037$; $f\left(-\frac{1}{\sqrt{2}}\right) \approx -0.43$; $f\left(\frac{1}{\sqrt{2}}\right) \approx 0.43$

So $f\left(\frac{1}{\sqrt{2}}\right)$ is the absolute maximum

$f\left(-\frac{1}{\sqrt{2}}\right)$ " " " minimum.

Examples

2. Find the extreme values of $g(x) = (\sqrt[3]{x})^2$ in the interval $[-2, 2]$.

1. $g'(x) = \frac{2}{3} x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$

   g' is never 0 in $[-2, 2]$ but g is not differentiable at $x=0$ so $x=0$ is our critical pt.

2. $g(0) = 0$

3. $g(-2) = \sqrt[3]{4}$, $g(2) = \sqrt[3]{4}$

4. g has absolute maximum $\sqrt[3]{4}$ at $x=-2$ and at $x=2$;

   g has absolute minimum 0 at $x=0$.

Concavity: second derivative

We have seen that the sign of the first derivative of a function f gives information on whether the function f increases or decreases.
When the second derivative exists, its sign gives information on the concavity of the function.

Concavity

- If the graph of a function f lies **above** all of its tangents on an interval $[a, b]$ then f is called **concave upwards** on $[a, b]$.
- If the graph of a function f lies **below** all of its tangents on an interval $[a, b]x$ then f is called **concave downwards** on $[a, b]$.

Concavity: second derivative

Suppose that the second derivative of f exists for all x in some interval $[a, b]$. The following holds

Sign of second derivative and concavity

- If $f''(x) > 0$ for all x in $[a, b]$ then the graph of f is concave upwards on $[a, b]$
- If $f''(x) < 0$ for all x in $[a, b]$ then the graph of f is concave downwards on $[a, b]$

Example. Let $q(x) = ax^2 + bx + c$ be a quadratic function.

$$q'(x) = 2ax + b \qquad \text{and} \qquad q''(x) = 2a.$$

The sign of the second derivative is exactly the sign of the parameter a. Indeed, we know that the graph of $q(x)$ is a parabola facing upwards if $a > 0$ and downwards if $a < 0$.

Quick check for local extreme values: second derivative test

The analysis we just did tells us that there is a quick way to find out if a critical point (at which first and second derivatives exist) is a local maximum, a local minimum or neither one. We need to assume that f'' is continuous near the critical point.

Second derivative test

- If $f'(c) = 0$ and $f''(c) > 0$ then $f(c)$ is a **local minimum** for f.
- If $f'(c) = 0$ and $f''(c) < 0$ then $f(c)$ is a **local maximum** for f.

Examples

1. Find and classify all local extreme values of the function $f(x) = x^3 - 3x^2 + 1$.

$f'(x) = 3x^2 - 6x$ f is differentiable everywhere

$\Rightarrow$ only local extreme values can occur at pts with derivative $= 0$

These are $x = 0$ and $x = 2$. We can use the 2nd derivative test:

$f''(x) = 6x - 6$ $\leadsto$ $f''(0) = -6 \Rightarrow f(0) = 1$ is a local max

$f''(2) = 6 \Rightarrow f(2) = -2$ is a local minimum

Examples

2. Find and classify all local extreme values of the function $h(x) = x^4 - 4x^3$.

$h'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$ is zero at $x=0$ and $x=3$

$h''(x) = 12x^2 - 24x = 12x(x-2)$ we can use the 2nd derivative test at $x=0$ and $x=3$

$h''(0) = 0$

$h''(3) = 36 > 0 \Rightarrow h(3) = -27$ is a local minimum.

→ here the 2nd derivative test doesn't help us!

Looking at the sign of the 1st derivative, however, we can deduce that $x=0$ is neither a maximum nor a minimum, as the function is decreasing both on the left and on the right of $x=0$.

When c is a critical point and $f''(c) = 0$ then the second derivative test doesn't help us. In these cases $f(c)$ can be a local maximum, a local minimum, or neither one. If (like in the case above) c is a critical point, $f''(c) = 0$ and the graph changes concavity at $x = c$ then we call $f(c)$ an **inflection point**.