MA133C & MA160
Calculus 1

Lecture 16

Applications of differentiation: optimisation

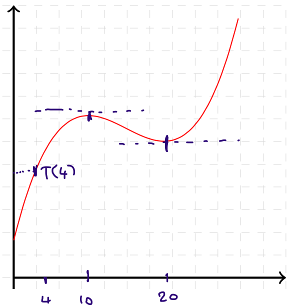
Suppose we want to book a flight ticket and we know that the price fluctuates with time according to the function:

$$T(x) = 50 + 200x - 15x^2 + \frac{x^3}{3}.$$

Also, suppose that we only have internet access to actually buy the tickets between $x = 4$ and $x = 24$. What is the most convenient moment for us to buy?
First of all, let's have a look at the graph of $T(x)$ to get an idea of the general behaviour of the price function. To sketch the graph, observe that:

- T is a cubic function
- $T(0) = 50$, $T(4) = 1894/3 \approx 631$, $T(24) = 818$
- $\lim_{x \to +\infty} T(x) = +\infty$
- $T'(x) = 200 - 30x + x^2$

Optimisation



$$T(x) = 50 + 200x - 15x^2 + \frac{x^3}{3}.$$
$$T'(x) = 200 - 30x + x^2 = (x-10)(x-20)$$

That is, the derivative T' is **positive** in $(-\infty, 10)$ and $(20, +\infty)$ and **negative** in $(10, 20)$.

This means: T increases for $x < 10$, then decreases for $10 < x < 20$, then increases again for $x > 20$.

What is an efficient way to find (if it exists) the value x for which the price is **minimal**?

As we can see from the graph, when T changes trend (i.e. it changes from increasing to decreasing, and from decreasing to increasing) its first derivative is 0.

Optimisation

$$T(x) = 50 + 200x - 15x^2 + \frac{x^3}{3}.$$
$$T'(x) = 200 - 30x + x^2 = (x-10)(x-20)$$

Our point of minimum is one of the following:

- one of the extremes of the interval, or
- a point at which the derivative is zero (negative to the left and positive to the right).

In our case the three candidates points of minimum are: $x = 4$, $x = 24$ and $x = 20$. All we need to do now is compare the three values of T at these points and choose x corresponding to the smallest:

$$T(4) = 1894/3 \approx 631; \qquad T(20) = 2150/3 \approx 717; \qquad T(24) = 818.$$

Optimisation

Conclusion: the most convenient time to buy our ticket is $x = 4$. We will call this a **point of minimum** for the function.

The corresponding price is $T(4) \approx 631$. We will call this the **minimum value** for the function T in the interval $[4, 24]$.

Note that the point $m = 20$ has the property that for all values of x "near m", the value of the function is greater than the value of the function at 20.

We will call the value $T(20)$ a **local minimum** for the function T.

Absolute maximum, absolute minimum

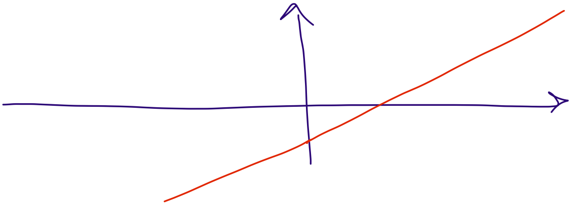
Let f be a function with domain D and let c be in D. We say that $f(c)$ is the

- **absolute** (or **global**) **maximum** value of f on D if $f(c) \geqslant f(x)$ for all x in D.
- **absolute** (or **global**) **minimum** value of f on D if $f(c) \leqslant f(x)$ for all x in D.

The absolute maximum and absolute minimum are also called **extreme values** of f.

Examples.

1. The function $f(x) = \dfrac{1}{3}x - 2$ defined on $\mathbb{R}$ has **no** absolute maximum and **no** absolute minimum.

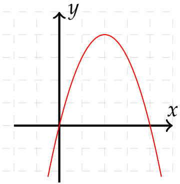


$\lim_{x \to +\infty} \frac{1}{3}x - 2 = +\infty$

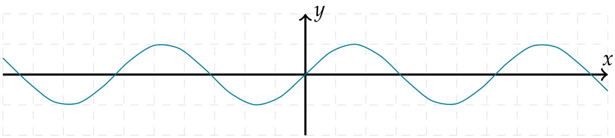
$\lim_{x \to -\infty} \frac{1}{3}x - 2 = -\infty$

Maximum and minimum values

2. The function $g(x) = 4x - x^2$ defined on $\mathbb{R}$ has absolute maximum value 4 at $x = 2$. It has **no** absolute minimum.



3. The function $h(x) = \sin(x)$ defined on $\mathbb{R}$ has absolute maximum value 1 and absolute minimum value -1. Both max and min are attained at **at infinitely many points** (all points of the form $x = \pi/2 + 2k\pi$ are points of maximum, all points of the form $x = -\pi/2 + 2k\pi$ are points of minimum).



Maximum and minimum values

Local maximum, local minimum

Let f be a function and let c be in its domain. We say that $f(c)$ is a

- **local maximum** value of f if $f(c) \geqslant f(x)$ for all x *near* c.
- **local minimum** value of f if $f(c) \leqslant f(x)$ for all x *near* c.

Local maxima and local minima are also called **local extreme values** of f.

Examples.

1. In our example from the beginning $T(x) = 50 + 200x - 15x^2 + \frac{x^3}{3}$':
   - $T(20)$ is a **local minimum**, and
   - $T(10)$ is a **local maximum**.
2. $f(x) = x^2 + 1$ has a local (and absolute) minimum 1 at $x = 0$.

Extreme Value Theorem

We have seen that not all functions admit extreme values. If the region in which we look for these extreme values is **confined** and the function **behaves well** in this region, then the extreme value theorem tells us that the function attains an absolute maximum and an absolute minimum. Formally:

Extreme value theorem*

If f is **continuous** on a **closed interval** $[a, b]$, then f attains an absolute maximum $f(x_M)$ and an absolute minimum $f(x_m)$ for some values x_m, x_M in the interval $[a, b]$.

Less formally, if we can draw the graph of a function f between a and b without picking up the pencil (that is, if the graph has no gaps, jumps, or vertical asymptotes in the closed interval $[a, b]$), then there exist (in the closed interval $[a, b]$) points at which the function f attains its maximum and its minimum.

Example

Let $f(x) = 3x^4 - 16x^3 + 18x^2$.

1. Find (if they exist) the extreme values of f on $\mathbb{R}$.
2. Find the extreme values of f in the interval $[-1, 4]$.

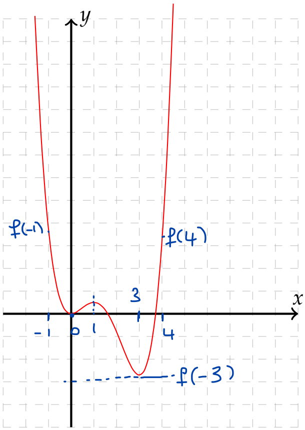
1. $\lim_{x \to \pm\infty} f(x) = +\infty \Rightarrow$ no ABS. MAX. on $\mathbb{R}$

$f'(x) = 12x^3 - 48x^2 + 36x = 12x(x-1)(x-3)$

3 pts with horizontal tangent: $x=0$, $x=1$, $x=3$

Together with the points $x=-1$, $x=4$, they are the candidates for our global max and min in the interval $[-1, 4]$.

Example



$$f(x) = 3x^4 - 16x^3 + 18x^2$$

- No absolute maximum on $\mathbb{R}$
- Absolute minimum $= -27$ at $x = 3$
- The Extreme Value Theorem tells us that the extreme values for f in the interval $[-1, 4]$ exist. These are:
  - Min $= -27$ attained at $x = 3$
  - Max $= 37$ attained at $x = -1$.

The function has local extreme values at $x = 0$ (loc. min), $x = 1$ (loc. max), $x = 3$ (loc. min).