MA133C & MA160
Calculus 1

Lecture 15

Recap

- Inverse functions
  - A function f is called a **one-to-one** function if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.
  - The **inverse** function f^{-1} of a one-to-one function f has domain the range of f and is defined, for any y in the range of f as $f^{-1}(y) = x$ if and only if $f(x) = y$.
  - The graph of f^{-1} is obtained by **reflecting** the graph of f about the line $y = x$.
- Logarithms
  - The **natural logarithm** is the inverse function of the exponential function: for a positive real number x
    $$\ln x = y \qquad \text{if} \qquad e^y = x.$$
  - $$\boxed{\begin{aligned} \ln(e^x) &= x && \text{for all real numbers } x\text{, and} \qquad (1) \\ e^{\ln x} &= x && \text{for all \textbf{positive} real numbers } x. \qquad (2) \end{aligned}}$$
  - $$\boxed{\frac{d}{dx}\ln x = \frac{1}{x}}$$

Newton's Law of Cooling*

Newton's Law of cooling states that the **rate of cooling** of an object is **proportional to the temperature difference** between the object and its surroindings–provided that this difference is not too large.

If we let $T(t)$ be the temperature of our cup of coffee at time t and T_r be the temperature of the room, then Newton's Law of cooling says that

$$\frac{dT}{dt} = k(T - T_r)$$

Let's call $f(t) := T(t) - T_r$. Then, by the rules of derivatives

$$\frac{df}{dt} = \frac{dT}{dt} = k(T - T_r) = kf(t).$$

This new function f has first derivative which is a multiple of the function itself. The exponential function has this property for a particular choice of exponent!

$$\frac{d}{dt}(e^{kt}) = ke^{kt}$$

Problems, 1

That is, Newton's Law of Cooling tells us that the temperature of an object in a colder (or warmer) environment is modelled by

$$T(t) = T_r + (T_0 - T_r)e^{kt}$$

for a suitable choice of k.

Problem. A warm (94°) cup of coffee is placed on the desk of a lecture hall where the temperature is 18°. After 10 min (recap+explanation of Newton's Law of Cooling) the coffee has cooled to 75°.

1. How long does it take for the coffee to cool to 65°?
2. What would be the temperature at the end of the lecture?
3. Write an expression for the time as a function of the temperature of the coffee.

Problems, 1

For us: $T_0 = 94°$, $T_r = 18°$, $T(10) = 75$:

$T(t) = 18 + 76e^{kt}$. We use ↗ to get a value for the constant k:

$T(10) = 18 + 76e^{10k}$ should be 75. Therefore

$76e^{10k} = 75 - 18 = 57$ Dividing both sides by 76 and taking natural log:

$10k = \ln\left(\frac{57}{76}\right)$, so $k \approx -0.029$

- To get t such that $T(t) = 65$ one solves $18 + 76e^{-0.029t} = 65$.
- To get t as a function of T we need to invert the function $T(t)$:

write $y = 18 + 76e^{kt}$, we have $kt = \ln\left(\frac{y-18}{76}\right)$, so $t = \frac{1}{k}\ln\left(\frac{y-18}{76}\right)$

Therefore $t(T) = \frac{1}{k}\ln\left(\frac{T-18}{76}\right)$ is our function.

Problems, 2

The cost of producing x metres of a certain fabric is

$$C(x) = 150 + 200x - 15x^2 + \frac{x^3}{3}.$$

- Find the **marginal cost*** function.
- Find $C'(25)$ and explain its meaning. What does it predict?
- Compare $C'(25)$ with the cost of manufacturing the 26st metre of fabric.

$*$In Economics, the "marginal cost function" is the derivative of the function givinsg production cost per unit. This gives the rate at which costs are increasing with respect to the production level. See also [Section 3.7, Stewart].

Problems, 2

The marginal cost is $C'(x) = 200 - 30x + x^2$

Its value at 25 is $C'(25) = 75$

This value is close to the value

$$C(26) - C(25) \approx 85,$$

which represents the cost of producing the 26th metre of fabric.

Problems, 3

A turtle (f), Achilles (g) and the White Rabbit (h) participate in the 10km Wonderland Race. Their positions as functions of time are given by:

$$f(t) = \ln(1 + t^2), \qquad g(t) = t^2 + 3t, \qquad h(t) = te^t.$$

1. In what order will the contestants arrive?
2. Who is leading after one minute?
3. A prize is also given to the contestant with the
   (a) ▸ Highest average velocity in the first minute
   (b) ▸ Highest instant velocity at minute 1
   (c) ▸ Lowest acceleration at minute 2

   who wins these?
4. Alice wants to take pictures of all contestants, between minutes 1 and 2. She knows that the best photo can be taken when the contestant is running at the lowest velocity. When should she take the picture of each of the contestants?

Problems, 3

1. is the same as: find t_1 such that $f(t_1) = 10$
t_2 " " $g(t_2) = 10$
t_3 " " $h(t_3) = 10$

which of t_1, t_2, t_3 is smaller? (it is t_3, the white rabbit wins).

2. On the other hand: $f(1) = \ln 2 < 1$
$g(1) = 4$
$h(1) = e$

- So Achilles leads at minute 1. He also has the highest average velocity in the interval $[0,1]$.
- For 3(b) compare $f'(1), g'(1)$ & $h'(1)$ and take the max

for 3(c) compare $f''(2), g''(2)$ and $h''(2)$ and take the min.