MA133C & MA160
Calculus 1

Lecture 14

Graph of the inverse function

Graph of f^{-1}

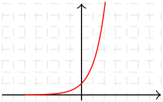
The graph of f^{-1} is obtained by **reflecting** the graph of f about the line $y = x$.

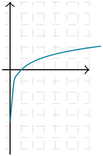
Alternatively, you can use the following criterion:

Graph of f^{-1}

The point (a, b) belongs to the graph of f if and only if the point (b, a) belongs to the graph of f^{-1}.

Example $f(x) = e^x$ is one-to-one. It has therefore an inverse function.





We will call this function the **natural logarithm** of x

The natural logarithm

The natural lograrithm is defined to answer the question: what power of e should we take to get some number a?

The natural logarithm

For a positive number x we call **natural logarithm of x** the number y such that $e^y = x$. We write

$$\ln x = y \qquad \text{if} \qquad e^y = x.$$

In particular

$$\boxed{\begin{aligned} \ln(e^x) &= x && \text{for all real numbers } x\text{, and} && (1)\\ e^{\ln x} &= x && \text{for all \textbf{positive} real numbers } x. && (2)\end{aligned}}$$

Note that, in particular, $\ln e = 1$.

Examples.

1. Find x if $\ln x = 3$. To find the solution you can either apply the definition, or you can consider the equation

$$\ln x = 3$$

and "apply" the exponential function to both sides, getting:

$$e^{\ln x} = e^3$$

Use then (1) from the previous slide to get $x = e^3$.

2. (From yesterday's "rumour" exercise)
Find t such that $\dfrac{1}{1+9e^{-t/2}} = \dfrac{1}{2}$. Here:

$$\frac{1}{1+9e^{-t/2}} = \frac{1}{2} \iff 1+9e^{-t/2} = 2 \iff 9e^{-t/2} = 1 \iff e^{-t/2} = \frac{1}{9}$$

Taking the natural logarithm on both sides we get $-\frac{t}{2} = \ln\left(\frac{1}{9}\right)$, that is
$t = -2\ln\left(\frac{1}{9}\right)$.

Other bases

In general, logarithmic functions are defined to solve exponential equations with **any base** $a > 0, a \neq 1$.

Logarithmic functions

Given $a > 0, a \neq 1$, for a positive number x we call **logarithm of x in base a** the number y such that $a^y = x$. We write

$$\log_a x = y \qquad \text{if} \qquad a^y = x.$$

In particular

$\log_a(a^x) = x$	for all real numbers x, and
$a^{\log_a x} = x$	for all **positive** real numbers x.

Note that $\log_a a = 1$.

Example. Calculate $\log_3 81$.

As $3^4 = 81$ we have $\log_3 81 = 4$.

Laws of logarithms

The following laws hold for any positive numbers x and y:

$$\log_a(xy) = \log_a x + \log_a y$$
$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$
$$\log_a(x^r) = r\log_a x \text{ for any real number } r$$

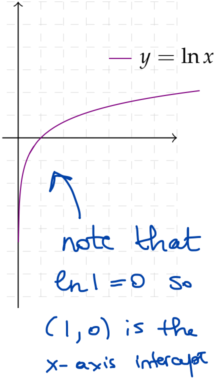
They hold in particular for the natural logarithm, that is when $a = e$.

Finally, the following **change of base formula** holds:

$$\log_a x = \frac{\ln x}{\ln a}.$$

Natural logarithm as a function

The natural logarithm is in particular the inverse function of the exponential function $f(x) = e^x$. As such, we know how its graph looks like:



- The natural logarithm is a function defined on $(0, +\infty)$.
- Its range is $\mathbb{R}$.
- It is increasing everywhere in its domain.
- However, it grows **very slowly**, slower than any power of x.
- The y-axis is a right vertical asymptote for this function. That is

$$\lim_{x \to 0^+} \ln x = -\infty.$$

Derivative of the natural logarithm

The natural logarithm turns out to be differentiable at all points of its domain. The following holds:

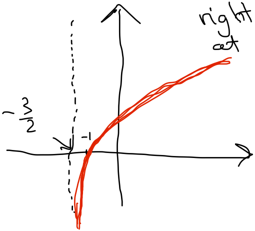
$$\boxed{\frac{d}{dx}\ln x = \frac{1}{x}}$$

Examples

1. Find the natural domain of $f(x) = \ln(2x+3)$, sketch its graph and compute its first derivative.

The natural logarithm is defined when its argument is strictly positive:

$2x+3>0$ which implies that domain of f is $(-\frac{3}{2}, +\infty)$



right vertical asymptote at $x=-\frac{3}{2}$. Also $f(-1)=0$

Finally, by applying the chain rule we get:

$$f'(x) = \frac{1}{2x+3}\cdot 2 = \frac{2}{2x+3}$$

Examples

2. If $g(x) = \dfrac{\ln x}{x^2}$ find $g'(1)$.

We will use the quotient rule:

$$g'(x) = \frac{\frac{1}{x}\cdot x^2 - 2x \cdot \ln x}{x^4} = \frac{x - 2x\ln x}{x^4} = \frac{\cancel{x}(1-2\ln x)}{x^{\cancel{4}3}} = \frac{1-2\ln x}{x^3}$$

So $g'(1) = \dfrac{1-2\ln(1)}{1} = 1$

3. Find the equation of the tangent to the graph of $u(x) = \ln\left(\dfrac{x^2+3}{2x+2}\right)$ at $x = 1$.

A good idea is to first use the laws of logarithms to rewrite $u(x)$:

$u(x) = \ln(x^2+3) - \ln(2x+2)$. Therefore, by applying the chain rule:

$u'(x) = \dfrac{2x}{x^2+3} - \dfrac{2}{2x+2}$. Its value at 1 is $u'(1) = \dfrac{2}{4} - \dfrac{1}{2} = 0$

Also, $u(1) = \ln\left(\dfrac{1+3}{2+2}\right) = \ln 1 = 0$

So the tangent at $x=1$ is the horizontal line $y=0$.

Derivatives of inverse functions*

In general, if f is an invertible function with inverse f^{-1} then

$$(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x.$$

This leads to a formula for the **derivative of the inverse function**, knowing that of f:

$$(f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}.$$

Examples.

1. $f(x) = e^x$. Then $f'(x) = e^x$ and $f^{-1}(x) = \ln x$. Let's apply the formula for the derivative of the inverse of f at x:

$$(f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))} = \frac{1}{e^{f^{-1}(x)}} = \frac{1}{e^{\ln(x)}} = \frac{1}{x}.$$

2. $f(x) = x^3$.