

MA133C & MA160
Calculus 1

Lecture 13



Announcements

- ▶ The new assignment is online, the deadline is **Tuesday, November 6**.
- ▶ Results of old sheets can be obtained by fetching again the relevant sheet.
- ▶ Student feedback (forms to be returned to the School of Maths).
- ▶ The last 2 (or more) lectures will be revision lectures.

Recap on differentiation by rule

► Derivatives of exponential and trigonometric functions:

$$\frac{d}{dx}(\sin(x)) = \cos(x); \quad \frac{d}{dx}(\cos(x)) = -\sin(x); \quad \frac{d}{dx}(\tan(x)) = \frac{1}{(\cos(x))^2}$$

$$\frac{d}{dx}e^x = e^x$$

► Product and quotient rules

$$(fg)' = f'g + fg'; \quad \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

Example: if $u(x) = x^2e^x$ then $u'(x) = (2x + x^2)e^x$;

if $v(x) = \frac{1+x^2}{1-x}$ then $v'(x) = \frac{-x^2 + 2x + 1}{(1-x)^2}$.

► Chain rule

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

Exercise [cf. Stewart 3.4.84]

Under certain circumstances a rumour spreads according to the equation

$$p(t) = \frac{1}{1 + ae^{-kt}},$$

where $p(t)$ is the proportion of the population that knows the rumour at time t and a, k are positive constants. In particular, let $a = 9, k = 0.5$.

- ▶ Find the rate of spread of the rumour at time t .
- ▶ Find $\lim_{t \rightarrow \infty} p(t)$.
- ▶ If t is measured in hours, how long does it take for 10% of the population to hear the rumour? How long for half of the population?

Exercise

$$p(t) = \frac{1}{1 + 9e^{-t/2}}$$

consider first $u(x) = e^{-\frac{x}{2}}$. Then by the chain rule
 $u'(x) = -\frac{1}{2}e^{-\frac{x}{2}}$

$$p'(t) = \frac{-(1 + 9e^{-t/2})'}{(1 + 9e^{-t/2})^2} = \frac{-9 \cdot (-\frac{1}{2})e^{-t/2}}{(1 + 9e^{-t/2})^2}$$

$$\boxed{\lim_{x \rightarrow \infty} e^{-x} = 0}$$

$$\lim_{t \rightarrow \infty} p(t) = \lim_{t \rightarrow \infty} \frac{1}{1 + 9e^{-t/2}} \stackrel{\text{limit rules}}{=} \frac{1}{\lim_{t \rightarrow \infty} (1 + 9e^{-t/2})} \stackrel{\text{limit rules}}{=} \frac{1}{1 + 9 \lim_{t \rightarrow \infty} e^{-t/2}} = \frac{1}{1 + 9 \cdot 0} = 1$$

The last question is the same as asking: find t such that $p(t) = \frac{1}{10}$
and check that $p(t) = \frac{1}{2}$. For the first:

$$\frac{1}{1 + 9e^{-t/2}} = \frac{1}{10} \Leftrightarrow 1 + 9e^{-t/2} = 10 \Leftrightarrow 9e^{-t/2} = 9 \Leftrightarrow e^{-t/2} = 1$$

whose solution is $t = 0$.

For the other we will need "the inverse" of the exponential function.

Inverting functions

In the previous example, the last question gave us a way to “change perspective” on the function we were analysing: knowing the time as a function of the proportion of the population who knows the rumour.

This change of perspective corresponds to taking the **inverse** of the relevant function. Let's consider some other examples.

- ▶ A particle moves on a straight line with velocity (as a function of time) given by $v(t) = (t - 2)^2$. At what time(s) does the particle stop? At what time(s) does it have velocity equal to 1?

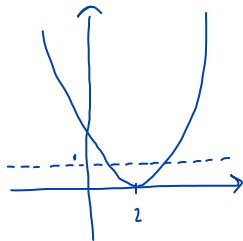
The particle stops when $v(t) = 0$, namely:

$$v(t) = 0 \Leftrightarrow (t-2)^2 = 0 \Leftrightarrow t = 2$$

The particle has velocity 1 when $v(t) = 1$, namely:

$$(t-2)^2 = 1 \Leftrightarrow t^2 - 4t + 4 - 1 = 0 \Leftrightarrow t^2 - 4t + 3 = 0$$

$$\Leftrightarrow (t-1)(t-3) = 0 \text{ that is when } t = 1 \text{ or } t = 3$$



Inverting functions

- ▶ As dry air moves upward, it expands and cools. The temperature as a function of the height appears to be $T(h) = 25 - 10h$ degrees. If we are on a hot air balloon with no altimeter but with a thermometer, how do we determine our height?

write

$$T = 25 - 10h \quad \text{and "solve for } h \text{" :}$$

$$10h = 25 - T, \quad \text{that is} \quad h = \frac{25 - T}{10} \quad \text{as a function of } T :$$

$$h(T) = \frac{25 - T}{10}$$

One-to-one functions

The function giving the temperature as a function of the height (and conversely) is a special type of function: a function which never takes on the same value twice.

One-to-one function

A function f is called a **one-to-one** function if

$$f(x_1) \neq f(x_2) \quad \text{whenever } x_1 \neq x_2.$$

As we saw in the examples before, a useful geometric method to determine whether a function is one-to-one or not is the following.

Horizontal line test

A function f is one-to-one if and only if no horizontal line intersects its graph more than once.

Inverse function

One-to-one functions can be “inverted”.

Inverse function

Let f be a one-to-one function. Then its **inverse** function f^{-1} has domain the range of f and is defined, for any y in the range of f as

$$f^{-1}(y) = x \quad \text{if and only if} \quad f(x) = y.$$

Example Let $f(x) = 3x - 2$. Then $f(x) = y$ if and only if $3x - 2 = y$, which yields $x = \frac{y + 2}{3}$.

This means: $f^{-1}(x) = \frac{1}{3}(x + 2)$.

Note that, in general for a one-to-one function f :

- ▶ the **domain** of $f^{-1} =$ range of f ;
- ▶ the **range** of $f^{-1} =$ domain of f .

Example

Find the inverse function of $f(x) = x^3 + 2$.

$$x^3 + 2 = y$$

$$x^3 = y - 2$$

$$x = \sqrt[3]{y - 2}$$

$$\text{So: } f^{-1}(x) = \sqrt[3]{x - 2}$$

- write $f(x) = y$
- solve for x
- write $f^{-1}(x)$ for x
and x for y

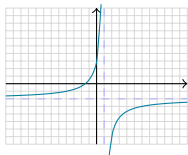
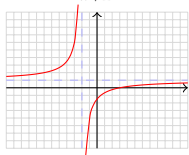
Graph of the inverse function

Graph of f^{-1}

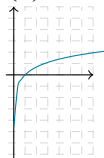
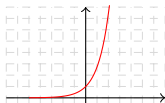
The graph of f^{-1} is obtained by **reflecting** the graph of f about the line $y = x$.

Examples

- $f(x) = \frac{x-3}{x+2} \rightsquigarrow f^{-1}(x) = \frac{2x+3}{1-x}$ with graphs:



- $f(x) = e^x$ is also one-to-one.. $\rightsquigarrow f^{-1}(x) = ??$



We will call this function the **natural logarithm** of x