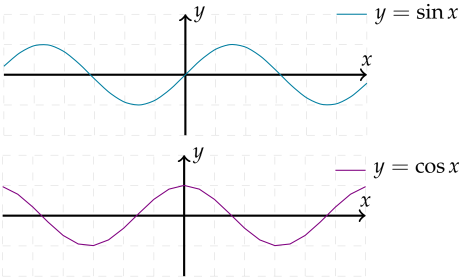
MA133C & MA160
Calculus 1

Lecture 12

Trigonometric functions: derivatives and special limits

Recall the functions sine and cosine and their graphs:



What are the derivatives of sine and cosine?

Note:

- Sine increases in the intervals in which cosine is positive, and decreases in the intervals in which cosine is negative
- Points with horizontal tangent to sine correspond to zeros of cosine

Trigonometric functions: derivatives and special limits

Let's first consider $f(x) = \sin(x)$ and let's apply the definition:

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} = \\
&= \sin(x) \lim_{h\to 0} \frac{\cos(h) - 1}{h} + \cos(x) \lim_{h\to 0} \frac{\sin(h)}{h} = \\
&= \cos(x)
\end{aligned}
$$

Where we used the following **special limits**:

- $\boxed{\lim_{\theta\to 0} \frac{\cos(\theta) - 1}{\theta} = 0}$ and
- $\boxed{\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1}$

Trigonometric functions: derivatives

The functions sine and cosine have derivatives:

- $(\sin(x))' = \cos(x)$
- $(\cos(x))' = -\sin(x)$

We can differentiate the tangent function by applying the quotient rule:

$$\begin{aligned}\frac{d}{dx}(\tan(x)) = \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right) &= \frac{\cos(x)\frac{d}{dx}(\sin(x)) - \sin(x)\frac{d}{dx}(\cos(x))}{(\cos(x))^2} \\ &= \frac{\cos(x)\cdot\cos(x) - \sin(x)\cdot(-\sin(x))}{(\cos(x))^2} \\ &= \frac{1}{(\cos(x))^2}\end{aligned}$$

That is

- $(\tan(x))' = \dfrac{1}{\cos^2(x)}$

Examples

1. Evaluate the following limit:

$$\lim_{\theta \to 0} \frac{\sin 4\theta}{\sin 3\theta}$$

2. Compute the derivative of $g(x) = x^2 \sin(x) \cos(x)$. (From exam paper 18/19)

Examples

1. Find an equation of the tangent line to the curve $y = 3x + 6\cos x$ at its point with x-coordinate $\pi/3$.

2. A mass on a spring vibrates horizontally on a smooth level surface. Its equation of motion is $x(t) = 8\sin t$, where t is in seconds and x in centimeters.
   - **(a)** Find the velocity and **acceleration** at time t.
   - **(b)** Find position, velocity and acceleration of the mass at time $t = 2\pi/3$. In what direction is it moving at that time?

Examples

A ladder 6m long rests against a vertical wall. Let θ be the angle between the top of the ladder and the wall and let x be the distance from the bottom of the ladder to the wall. If the bottom of the ladder slides away from the wall, how fast does x change with respect to θ when $\theta = \pi/3$?

Derivatives of compositions of functions: the chain rule

To compute the derivative of the composition of two functions f and g we use the **chain rule**:

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x),$$

that is, the derivative of f composed with g at x is the product of f' evaluated at $g(x)$ with g' calculated at x.

The function $f \circ g$ is differentiable at a point x if g is differentiable at x and f is differentiable at $g(x)$.

Example. Compute the derivative of the function $u(x) = \sin(x^2 + 3x - 1)$.

First, we identify the two functions whose composition gives u: if $g(x) = x^2 + 3x - 1$ and $f(x) = \sin(x)$ then $u = f \circ g$. We can now apply the chain rule:

$$u'(x) = f'(g(x))g'(x) = (\cos(x^2 + 3x - 1)) \cdot (2x + 3).$$

Examples

- Compute the derivative of $f(x) = e^{x^2+1} + 1$.

- Compute the derivative of $g(x) = \sqrt{x^3 + 2x - 1}$.

- Find an equation of the tangent line to the curve $y = \dfrac{2}{1 + e^{-x}}$ at each of its points with y-coordinate equal to 1.