MA133C & MA160
Calculus 1

Lecture 11

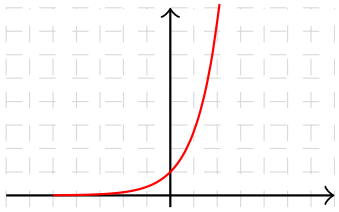
Recap

- The derivative of a function f at a point a is the limit of the average rate of change:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

- The number $f'(a)$ is the **slope of the tangent** to the graph of f at $(a, f(a))$.
- If f is a function modelling the position as a function of time then f' is the function giving the instant velocity of the motion.
- We have given simple computational rules for derivatives of powers, polynomials, sums and differences of functions.
- The **sign** of the derivative is related to the behaviour of the function:
  - if the derivative f' is **positive** in some interval then the function f is **increasing** in that interval
  - if the derivative f' is **negative** in some interval then the function f is **decreasing** in that interval
- Puzzle!

Exponential: special limit and derivative



Recall the exponential function $f(x) = e^x$ with graph

What is the derivative of this function?

As the exponential function is an increasing function we expect its derivative to be non-negative.

Let's apply the definition:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = e^x \lim_{h\to 0} \frac{e^h - 1}{h} = e^x$$

Facts:

- $\boxed{\lim_{x\to 0} \frac{e^x - 1}{x} = 1}$
- $\boxed{\frac{\mathrm{d}}{\mathrm{dx}} e^x = e^x.}$

Derivatives of exponential function: examples

Examples.

(1) Compute the derivative of $f(x) = 2x^2 - e^x + 100$.

(2) Show that the curve $y = 2e^x + 3x + 5x^3$ has no tangent line with slope 2.

Differentiation by rule: products

The product of two differentiable functions f and g has derivative:

$$\boxed{(fg)' = f'g + fg'}$$

Examples.

- Compute the derivative of $u(x) = xe^{x+2}$.
  First, we rewrite $u(x) = e^2 \cdot x \cdot e^x$. Therefore

$$u'(x) = e^2\left((x)' \cdot e^x + x \cdot (e^x)'\right) = e^2(1 \cdot e^x + x \cdot e^x) = e^{x+2}(1+x)$$

- Compute the derivative of $v(x) = \dfrac{e^{x-3}}{x^2}$.
  Again, let's rewrite v in a convenient way: $v(x) = \dfrac{1}{e^3} \cdot e^x \cdot \dfrac{1}{x^2}$. Therefore

$$v'(x) = \frac{1}{e^3}\left(e^x \cdot x^{-2} + e^x \cdot (-2x^{-3})\right) = \cdots = \frac{e^{x-3}}{x^2}\left(1 - \frac{2}{x}\right).$$

Differentiation by rule: quotients

The quotient of two differentiable functions f and g has derivative:

$$\boxed{\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}}$$

Examples.

- Compute the derivative of $u(x) = \dfrac{x^2 - 3x + 1}{x^3 - 1}$.

- Find an equation for the line that is tangent to the curve

$$y = \frac{1}{x^2 + 1}$$

  at the point $(-1, \frac{1}{2})$.

(From exam paper 18/19)