MA133C & MA160
Calculus 1

Lecture 10

Secants and tangents to graphs of functions

Slope of a secant

Given a function f and two points in its domain x and a, the **secant** through the points of the graph $P = (x, f(x))$ and $Q = (a, f(a))$ has slope

$$m_{PQ} = \frac{f(a) - f(x)}{a - x}.$$

The **tangent** to the graph of f at a point $P = (x, f(x))$ can be interpreted as the limit of the secants as Q approaches P (that is, as a approaches x. If we write $a = x + h$ we get the following.

Slope of the tangent at a point

The **tangent** to the graph $y = f(x)$ at the point $P = (x, f(x))$ has slope

$$m_P = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

The derivative as a function

The definition of the derivative of a function at a point implies that this is a law associating to a real value x another real value $f'(x)$. In other words, the derivative of a function is a function itself.

Derivative as a function

Given a function f and any number x for which the limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists, we assign to x the number $f'(x)$ defining in this way a new function f'.
We say that the function f is **differentiable** at x if $f'(x)$ exists.
The domain of the function f' is the set of values for which the above limit exists.

The value of the function f' at a point x can be interpreted geometrically as the slope of the tangent line to the graph of f at the point $(x, f(x))$.

Example

Let $f(x) = x^3 - x$.

(a) Find a formula for $f'(x)$.

(b) Compare the graphs of f and of f'.

Differentiation by rule

We noticed in our examples that computing the limits defining a derivative at different points shows some similarities. This suggests that there should be computational rules that we can use to determine the derivative of a function (in most cases...)

Derivatives of basic functions

- If $f(x) = c$ then $f'(x) = 0$ — The derivative of a constant is zero
- If $f(x) = x$ then $f'(x) = 1$ — The derivative of x is 1
- If $f(x) = x^n$ then $f'(x) = nx^{n-1}$ — Rule for the derivative of a power

Examples.

- The derivative of $f(x) = x^{\mathbf{4}}$ is $f'(x) = \mathbf{4}x^{\mathbf{4}-1} = 4x^3$.
- The derivative of $g(x) = \frac{1}{x^2} = x^{\mathbf{-2}}$ is $g'(x) = \mathbf{-2}x^{\mathbf{-2}-1} = -\frac{2}{x^3}$.

Differentiation by rule

New derivatives from old

- If c is a constant and f is differentiable then the derivative of $c \cdot f(x)$ is $c \cdot f'(x)$
- The derivative of the sum of two differentiable functions is the sum of the derivatives.
- The derivative of the difference of two differentiable functions is the difference of the derivatives.

Example Determine the derivative of $f(x) = 3x^2 + 5x + \frac{1}{x}$ (From exam paper 18/19)

Example

Find a cubic function whose graph has horizontal tangents at the points $(-2, 6)$ and $(2, 0)$.

a cubic function has the form: $f(x) = ax^3 + bx^2 + cx + d$ for some a, b, c, d.

To determine the coefficients we should use the infor from the text:

$$f(2) = 0 \quad , \quad f(-2) = 6$$

and

$$f'(-2) = 0 \, , \quad f'(2) = 0 . \qquad f'(x) = 3ax^2 + 2bx + c$$

These conditions translate to the linear system

$$\begin{cases} 8a + 4b + 2c + d = 0 \\ -8a + 4b - 2c + d = 6 \\ 12a + 4b + c = 0 \\ 12a - 4b + c = 0 \end{cases}$$