§ Eigenvalues & Eigenvectors:

Let A be a 2×2 matrix the we get a Linear Transformation

$A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ via

$: (x, y) \longrightarrow A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$

$= (ax + by, cx + dy)$

We Proved A sends lines to lines

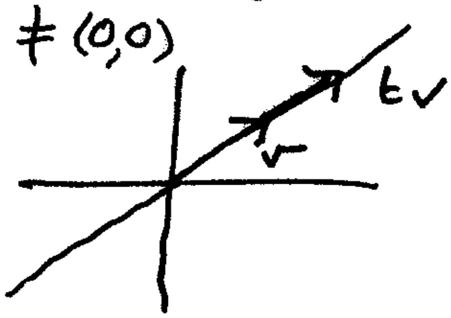
& " " sends lines through $(0,0)$ to lines through $(0,0)$

<u>Question</u>: Are there any lines through $(0,0)$ fixed ie sent to themselves?

let $\ell = \{ tv \mid t \in \mathbb{R} \}$

$(v \neq (0,0))$

So if $A(\ell) = \ell$



$\Rightarrow A(tv) = sv$ for some $s \in \mathbb{R}$

$\Rightarrow tA(v) = sv$

$\Rightarrow Av = \frac{s}{t} v$ $(t \neq 0)$

ie $Av = \lambda v$ $\lambda \in \mathbb{R}$.

So Defn: A a 2×2 matrix & $v \in \mathbb{R}^2$ a nonzero vector (ie $v \neq (0,0)$) If there exists $\lambda \in \mathbb{R}$ s.t $Av = \lambda v$, we call λ an eigenvalue with corresponding eigenvector v.

How to Find Them: If $Av = \lambda v$ $\quad v \neq (0,0)$, $\lambda \in \mathbb{R}$

$$\Rightarrow \quad Av = \lambda I v \quad \text{where } I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \quad (A - \lambda I)v = (0,0) \quad (\dagger)$$

But $\Rightarrow$ $(A-\lambda I)^{-1}$ doesn't exist since if it did multiply both side of $(\dagger)$ by it to get

$$v = \underbrace{(A-\lambda I)^{-1}}_{\text{a matrix}}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = (0,0)$$

But $v \neq (0,0)$ $\lightning$)

So $|A - \lambda I| = 0$ (called the characteristic eqn)

Can Solve for λ: e.g. $A = \begin{pmatrix} -1 & -2 \\ 4 & 5 \end{pmatrix}$

$$|A - \lambda I| = \left| \begin{pmatrix} -1 & -2 \\ 4 & 5 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right|$$

$$= \begin{vmatrix} -1-\lambda & -2 \\ 4 & 5-\lambda \end{vmatrix} = (-1-\lambda)(5-\lambda) + 8$$

$$= \lambda^2 - 4\lambda - 5 + 8$$

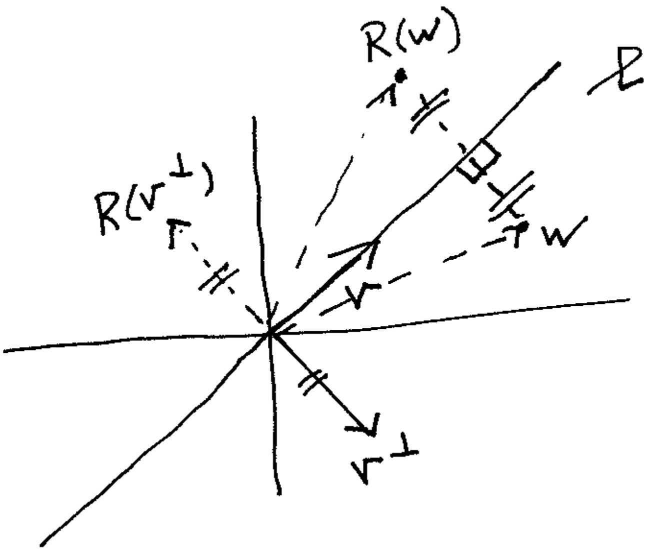
So $|A - \lambda I| = 0 \Rightarrow \lambda^2 - 4\lambda + 3 = 0$

$$(\lambda - 1)(\lambda - 3) = 0$$

So 2 possible eigenvalues $\lambda = 1, 3$

Examples: Note: a Rotation ↺ about the origin by an angle $\theta \neq n\pi$, $n \in \mathbb{N}$ doesn't fix any line ie send it to itself. So $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ has no eigen values or eigen vectors.

Ex: Let ℓ be a line through $(0,0) \in \mathbb{R}^2$ ie $y = mx$, $m \in \mathbb{R}$. Then $\ell = \{tv \mid t \in \mathbb{R}\}$



Let $R: \mathbb{R}^2 \to \mathbb{R}^2$ be reflection in ℓ

As $v \in \ell$

$R(v) = v$

$\Rightarrow \lambda_1 = 1$ is ~~the~~ eigen value with eigen vector v

also if $v^\perp$ is perpendicular to v then

$R(v^\perp) = -v^\perp$ ie $\lambda_2 = -1$ is an eigen value of R with eigen vector $v^\perp$.

So a Reflection in ℓ has $\lambda_i = \pm 1$
$i = 1, 2$

Recall from last day: Given A a 2×2 matrix a vector $v \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ s.t.

$$Av = \lambda v \qquad \lambda \in \mathbb{R}$$ is called an eigenvector (e-vector) with corresponding eigenvalue (e-value) λ.

How to find possible values of λ

Solve the (characteristic) equation

$$|A - \lambda I| = 0 \quad \text{(a quadratic in } \lambda\text{)}$$

Ex: $A = \begin{pmatrix} -1 & -2 \\ 4 & 5 \end{pmatrix} \qquad \lambda I = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$

$$|A - \lambda I| = \begin{vmatrix} -1-\lambda & -2 \\ 4 & 5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-1-\lambda)(5-\lambda) + 8 = 0$$

$$\Rightarrow \lambda^2 - 4\lambda + 3 = 0$$

$$\Rightarrow (\lambda - 1)(\lambda - 3) = 0$$

$$\Rightarrow \lambda = 1, 3$$

Now Find the corresponding e-vector for each e-value:

$\lambda_1 = 1$. So solve $Av_1 = \lambda_1 v_1$ for v_1

$$\iff (A - \lambda_1 I)v_1 = \begin{pmatrix}0\\0\end{pmatrix}$$

ie $\begin{pmatrix}-2 & -2\\ 4 & 4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$ $(\lambda_1 = 1)$

$$\Rightarrow -2x - 2y = 0$$

$$(4x + 4y = 0)$$

ie $x = -y$ e.g. $v_1 = \begin{pmatrix}-1\\1\end{pmatrix}$

(check: is $A\begin{pmatrix}-1\\1\end{pmatrix} = \underbrace{1}_{=\lambda_1}\begin{pmatrix}-1\\1\end{pmatrix}$

$$\begin{pmatrix}-1 & -2\\ 4 & 5\end{pmatrix}\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix} \checkmark)$$

Now Find v_2 s.t. $Av_2 = \lambda_2 v_2$

$(\lambda_2 = 3)$:

ie Solve $(A - 3I)v_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

for $v_2 = \begin{pmatrix} x \\ y \end{pmatrix}$

ie $\begin{pmatrix} -4 & -2 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

ie $-4x - 2y = 0$

$2x = -y \quad y = -2x$

e.g $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Check: $A\begin{pmatrix} 1 \\ -2 \end{pmatrix} \stackrel{?}{=} 3\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\begin{pmatrix} -1 & -2 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \end{pmatrix} = 3\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ ✓

Ex: $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ Find e-values & e-vectors:

Step I: solve for λ:

$$|A - \lambda I| = 0$$

ie $\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0$

ie $(1-\lambda)^2 - 4 = 0$

ie $\lambda^2 - 2\lambda - 3 = 0$

ie $(\lambda+1)(\lambda-3) = 0$

So $\lambda_1 = -1 \quad \lambda_2 = 3$.

Step 2: If each λ_i $(i = 1, 2)$
Solve $(A - \lambda_i I) v_i = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
for v_i the e-vectors.

$\lambda_1 = -1$ ie Solve

$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{v_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left((A - (-1)I)\, v_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right)$$

ie $2x + 2y = 0$

ie $x = -y$

eg. $v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\lambda_2 = 3$ ie Solve

$$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{v_2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

ie $-2x + 2y = 0 \Rightarrow x = y$

eg. $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Application: If know A has e-values λ_1 & λ_2 with corresponding e-vectors v_1 & v_2

ie $Av_1 = \lambda_1 v_1$ & $Av_2 = \lambda_2 v_2$

Let $E := \begin{pmatrix} \uparrow & \uparrow \\ v_1 & v_2 \\ \downarrow & \downarrow \end{pmatrix}$ & $D := \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

$$AE = \begin{pmatrix} \uparrow & \uparrow \\ \lambda_1 v_1 & \lambda_2 v_2 \\ \downarrow & \downarrow \end{pmatrix} = ED$$

ie $E^{-1}AE = D \implies$

$$E^{-1}A\underbrace{EE^{-1}}AE \cdots \underbrace{EE^{-1}}AE = D^n = \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}$$

$$\| $$

$$E^{-1}A^nE = \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}$$

ie $\boxed{A^n = E \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} E^{-1}}$