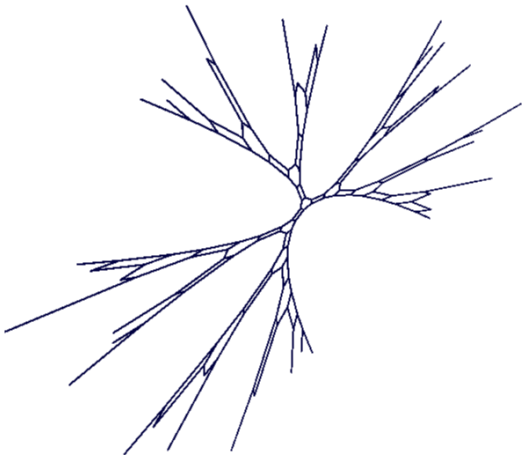
Topology

beyond a first syllabus



by NUIG undergraduates

2020

Preface

Each MA342 student was required to contribute a co-authored article to this volume on *Topology: Beyond a First Syllabus*. Their contribution counts for 15% of the module assessment.

Submission details

1. Each article should be no more than five pages maximum (including bibliography), should be written on a topology topic not covered in the MA342 exam syllabus, and should be written with a view to helping other MA342 students broaden their understanding of topology. The intended readership is MA342 students, and not professors of topology!
2. Each article should have at least two and no more than three co-authors.

   All co-authors will receive a common score for their contribution.
3. The article should be submitted as a pdf file produced using the Latex style file for the Proceedings of the American Mathematical Society. Please do not adjust the default page size and font size. The syle file is available in Overleaf.
4. You must follow all the AMS guidelines, and in particular:
   (a) Choose the most appropriate communicating AMS editor.
   (b) Include an abstract.
   (c) Include a primary Mathematical Subject Classification, and a secondary classification if appropriate.
   (d) Include a bibliography, with each bibliographic entry being cited at least once in the body of the article.

Possible topics

The MA342 exam syllabus is defined by the problem sheet, the lectures, and recent past exams. You are free to write on anything that complements, or expands on, this syllabus and that will help to improve other MA342 students' general understanding of the area of topology. You could choose an interesting definition and illustrate it and its use. You could choose a theorem and illustrate what it says. You could state a theorem and present a proof. You could write about some aspect of the life of a topologist. You could write about a recent trend in topology. The possibilities are endless. A few specific ideas are listed below.

- Discuss the quote:

  > Topology! The stratosphere of human thought! In the twenty-fourth century it might just possibly be of use to somebody, but for the present ...
  > — Solzhenitsyn, In the First Circle

- Explain/illustrate what the Poincaré Conjecture says, and write about the history of its proof. A statement of the conjecture, which is now a theorem, can be found on the Wikipedia topology page.
- State and prove any theorem from the later chapters of M.A. Armstrong's book *Basic Topology*.
- State and illustrate one of the topology theorems listed on the Wikipedia *Theorems in Topology* page.
- State and prove any theorem from Aisling McCluskey's book *Undergraduate Topology: A Working Textbook*
- What is a knot, what are the main goals of knot theory, and what were the origins of knot theory? (See for instance M.A. Armstrong's book *Basic Topology*.)
- Explain/illustrate what is meant by the homology of a simplicial complex. How does it relate to the Euler characteristic of a simplicial complex? (See for instance M.A. Armstrong's book *Basic Topology*.)
- Explain the Mapper clustering algorithm due to Singh, Mmoli, and Carlsson. Use a Google search to get started. If you are really ambitious you could even give an example using the R-package TDAmapper.
- Explain/illustrate what persistent homology is, and how it is used to understand point cloud data sets. Use a Google search to get started. If you are really ambitious you could even give an example using the R-package TDA.
- Write about topologists and the Fields medal. Which topologists have won the medal? Why did they win it? To get started you could look at the Wikipedia page on the Fields Medal.
- And so forth ...

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THE MAPPER ALGORITHM

DEARBHLA FITZPATRICK AND MEGAN TULLY

Abstract. In this paper we will demonstrate that by applying topological methods to complex high dimensional data we can extract shapes and in turn obtain insight about them. We will do so by explaining the fundamentals of the mapper algorithm. We will also include a real world application of this method to depict the advances it has made to Topological Data Analysis (TDA).

1. The use of topology in data analytics

1.1. **Shape.**

Shape is an organising principle for data analytics. Shape drives the choice of the model and is responsible for the effectiveness of the model we have chosen. For example, in figure 1.1.1 the data is linearly distributed. Therefore, it is most appropriate to fit a linear regression model. Fitting this model allows us to make predictions and describe the data in a simple way. However, if we were to apply the same model to the clustered data in Figure 1.1.2 it would perform very poorly. This demonstrates how shape defines how effective the model we choose will be.

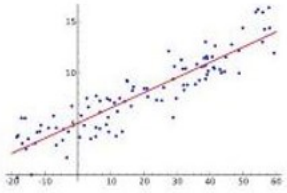

Figure 1.1.1

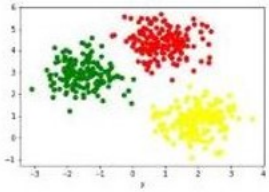

Figure 1.1.2

Linear Regression vs Clustering

1.2. **Why Topology?**

There are two mathematical areas that offer an understanding of shapes, Topology and Geometry. Topology does not care for distance but more so locality of shapes in Euclidean space. Therefore,Topology is a much more natural choice for data analysis as

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we are dealing with real world data and cannot, beyond a doubt, trust the scale of our data. Topology is useful when analysing high dimensional complex data due to the fact that the key topological features of a shape relies only on the intrinsic dimensions of the shape you are studying not on the dimension of space it is embedded.

2. Mapper clustering algorithm

2.1. **Point cloud data.**

Point cloud data are finite subsets of $\mathbb{R}^n$. Point clouds are generally produced by 3D scanners, which measure many points on the external surfaces of objects around them. This is demonstrated in Figure 2.1 where a hand is represented as point cloud data. It is often desirable to find images of various kinds attached to point cloud data which allow one to obtain a qualitative understanding of them through direct visualization. When we are developing such a representation it is important to think of properties that would make the imaging method most effective.

A Original Point Cloud

Figure 2.1.1

Below is a list of some of the desirable properties:

(1) Insensitivity to metric
Metrics used to analyse modern data sets are not derived from a particularly refined theory but are rather constructed as a reasonable quantitative proxy. Therefore, it would be preferable if the imaging method is, to some extent, insensitive to specified quantitative changes.
(2) Understanding sensitivity to parameter changes
It is desirable to find an imaging method that summaries the algorithms behaviour under any choice of parameters.
(3) Multiscale representations
We wish to understand point clouds at various resolutions and have the ability to provide output at various levels of comparison. This is also used to distinguish between actual features and artefacts. Actual features are more likely to appear at multiple scales.

The method discussed in the following sections addresses each of these points.

2.2. **Topological version of the mapper.**

The most important topological notion that the mapper method relies on is the nerve of a cover.

Definition 2.2.1 (Nerve of a cover). : Given a cover $C=(U_\alpha)\alpha\in A$ of X, we define the nerve of the cover C to be the simplicial complex $N(C)$whose vertex set is the index set A, and where a subset $(\alpha_0,\alpha_1,...,\alpha_k)\subseteq$ A spans a k-simplex in $N(C)$ if and only if $C_{\alpha 0}\cap C_{\alpha 1}\cap...\cap C_{\alpha k}\neq\emptyset$.

Definition 2.2.2 (Simplicial complex). : A simplicial complex K with a vertex set V is a collection of subsets of V with the condition that if $\sigma\in 2^V$ is in K, then all subsets of σ are in K.

What these definitions essentially tell us is that a nerve of an open covering is a construction of an abstract simplicial complex from an open covering of a topological space X.

How do we apply these definitions to the mapper?
Suppose we are given a base data space X that we wish to interpret. We apply a continuous function $f : X \to Z$, where Z is a space we understand well, such as $\mathbb{R}$. We choose a covering of Z and pull it back via f, then split each preimage into its connected components, finally taking as our output the nerve of this new covering of X. If we can contract each connected component, this nerve is homotopy equivalent to X. Even if this condition does not hold, the nerve still preserves some of the Topology of X

In the next section we will describe a method for transporting this construction from the setting of topological spaces to the setting of point clouds.

2.3. The statistical version of the mapper.

We must now develop a mapper that we can apply to point cloud data. The main idea involved in transitioning from the topological version to the statistical version is we alternatively use a clustering method to partition the data.

We must assume:

(1) The point cloud contains N points $x \in X$
(2) We have a function $f : X \to \mathbb{R}$ whose value is known for the N data points (This function is known as a filter)
(3) It is possible to compute inner-point distances between the points in the data.

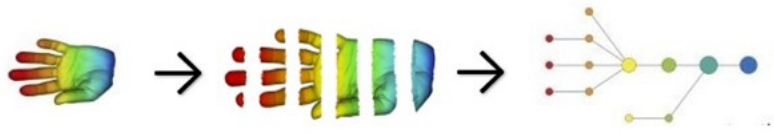

Figure 2.3.1 Figure 2.3.2 Figure 2.3.3

Figure 2.3

- In figure 2.3.1 we see a filter value is applied to the point cloud (seen in figure 2.1.1) and the object is now coloured by the values of the filter function.
- In figure 2.3.2 we see that the data is then binned into overlapping groups
- In figure 2.3.3 we see that each of the bins are clustered and a network is built.

This algorithm is closely comparable to local clustering as you must keep track of how the local clusters are related to each other. The edges are just as important as the nodes because they tell us where there are overlaps in the data clusters.

Each filter function generates different shaped networks which allows us to explore the data from different mathematical perspectives.
Some filters, when applied, may not produce anything of interest. This notion is discussed in an example in section 1.1.
We works with a data set experimentally to find values for which the network structure permits the identification of subgroups (such as the tips of flares, or clusters) of interest.

3. Applications

3.1. **Healthcare.**

An interesting application of the Mapper Algorithm is its use in healthcare. For example, Collaborator of Ayasdi, David Schneider from Stanford University applied the Mapper Algorithm to disease progression. He advised that you can be healthy and sick and that there is a continuum of states between healthy and sick. Looking at this data from a topological prospective using the Mapper Algorithm, Schneider realised that the model for disease progression is a loop. He came to this conclusion when he noticed that the path the disease takes as the subject is infected is not the same as the path to recovery. By mapping this loop, the path infected individuals pass through can be traced. This is done in a non-dimensional networks allowing the shape of the data to be interpreted more accessibly. This map identifies the current location of the host and predicts the future route of the infection. The nodes in graphing this data represents clusters of infected individuals and the edges connect nodes that have samples in common. The two parameters needed to map this data are the locality of the nodes and the filters. Each node is differentiated by different parameters displayed by infected individuals. The filter chosen to map disease progression is the density estimator. This method uses colour to show a low or high value of a pathogen in an infected individual.

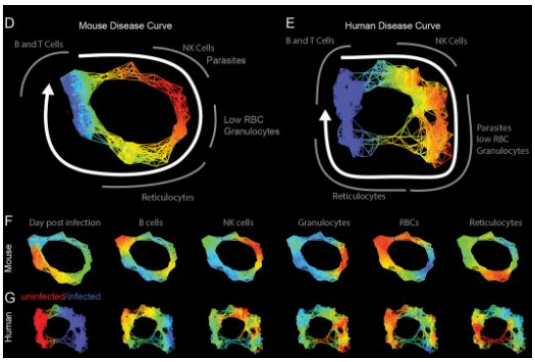

Figure 3.1

Example 3.1. A study was carried out to trace the path of a children infected with malaria. Mice aswell as children were used in this experiment as they both form the same looping dataset and their infections are collinear. Parasites, Red Blood Cells and Granulocytes were the parameters that were mapped. Figure 3 shows TDA maps of mice (D) and humans (E) infected with malaria. The white arrow shows the route of the disease progression with time. The density estimator uses a colour scheme in (D) and (E) that marks parasite density. Blue represents low values while red represents high.

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TOPOLOGICAL INSULATORS: AN APPLICATION OF TOPOLOGY

DARRAGH O'LOUGHLIN AND WILLIAM COMASKEY

(Communicated by)

Abstract. "Topology, the stratosphere of human thought. In the twenty fourth century it might possibly be of use to somebody. But for the present..." That was a quote from Solzhenitsyn, a famous Russian philosopher, where he jokes how topology will be of no use to anyone until a distant future. He might have overshot his estimate by a few hundred years as we could be on the brink of achieving such an application. Topological insulators despite their name have superconductive properties and being eyed as a replacement to the conductors used in every day devices such as your phone, laptop or smart watches. In this paper we are going to discuss topological insulators, what they are, how they work and why they're important.

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1. What is a Topological Insulator?

An insulator is a material that does not readily conduct electricity. A Topological Insulator is a material that behaves as an insulator in its interior but whose surface contains conducting states, meaning electrons can move along the surface of the material. But what makes topological insulators special is their superconductivity wherein electrical resistance vanishes.(1) These conductive surfaces or boundaries originate from topological invariants which never change as long as the material remains insulating.

Topology as we know studies the properties of objects that are invariant under smooth deformations. The unrivaled example being the deformation of a doughnut into a coffee cup. Insulators are topologically equivalent if they can be smoothly changed into each other (without ever becoming a conductor) and are topologically distinct if they cannot(2). What this means is that there are properties in insulators that are topological and admits the possibility that there are many species of insulator.

Unlike the doughnut/coffee cup pair, the trefoil knot and the closed loop have different topological invariants and thus neither can be deformed to become the other. In our case, the trefoil knot can represent topological insulator and the simple loop can represent an ordinary insulator. Since there is no continuous deformation where the knot can become the loop, there must be a surface where the string is cut. More formally the topological invariants cannot remain defined. If the topological invariants are always defined for an insulator, then the surface must be metallic.

2010 *Mathematics Subject Classification.* Primary .

So far the field of topological insulators has been focused on bismuth and antimony chalcogenide based materials. Bismuth is a brittle, crystalline, white metal with a slight pink tinge currently used in fire detectors and extinguishers and also in fuses and solders.(8)

2. How does a topological insulator work?

Imagine a 2-dimensional film of electrons with a large magnetic field pointing through it. Conduction occurs when electrons move from one place to another and when magnetic field gets large, this makes the electrons in the center of the film go around in circles and the film doesn't conduct electricity. But the electrons at the edge of the film are conducting electricity because they don't have the space to go around in circles and instead move around the edge of the film. This is called the "Hall Conductance". This conductance on the edge is topologically protected because no matter how you bend or squeeze the film, the film will remain conducting around its edges. This conducting is protected the same way the hole in the doughnut is protected.(5)

Say we have two of these films and put them on top of one another but with magnetic fields pointing outwards in opposite directions. Although this is not possible to have two magnetic fields behave this way, it is possible to have the electrons spin as if this did occur. This is what happens in topological insulators where we have one edge with up-spin electrons going one way and the down-spin electrons going the opposite way. and both are topologically protected.

In a typical conductor, electrical current flows everywhere. Insulators, on the other hand, do not readily conduct electricity. In topological insulators the interior works as an insulator, but the boundaries are conductive due to its topological property, resulting in the feature called 'topological edge conduction' or "Hall Conductance". Topological insulators have an energy gap in the bulk interior, just as in an ordinary insulator, but it contains conducting states localized on its surface.

Topology is the mathematical study of the properties of a geometric figure or solid that is unchanged by stretching or bending. Applying this concept to electronic materials has led to the discovery of many interesting phenomena, including topological edge conductance. Working like highways for electrons, channels of topological edge conduction allow electrons to travel with little or no resistance. Furthermore, because the edge channels can potentially be very narrow, electronic devices can be further miniaturized.

Several materials have been shown to be 3D topological insulators. 2D topological insulators are rare. Conduction appears at the surface of a 3D topological insulator; for a 2D film-like material however such conducting features exist solely at the edges of the film.

3. why are topological insulators important

Topological Insulators are hugely important for 2 reasons: Superconductivity and Energy Efficiency

3.1. **Superconductivity.** This property of topological insulators has applications in possibly any electronic device we possess. Every phone, computer and camera we have holds computer chips and transistors. These parts are all made of silicon

which was a revolution when it was first synthesised and it kick-started the information technology age we are currently in. But we are approaching a point that we will maximise silicons usefulness as it has limited conductive properties (i.e: it is electrically resistant). Topological Insulators hope to take up silicons mantle as the semiconductor of computers and micro-electric devices and propel us into a second technological age.

What separates topological insulators from other materials is its ability to both insulate and conduct at the same time. An application for such a property is Quantum Computing. Quantum Computing is the use of quantum-mechanical phenomena such as superposition and entanglement to perform computation. A quantum computer is used to perform such computation. In quantum computers there are qubits which are the quantum computer equivalent of bits in a classical computer. In a classical computer, bits may take one of many states, such as ones and zeroes, at any given time, which allows for the transmission of information in distinct forms. These forms are finite possibilities of finite states, bound by the laws of classical physics. The quantum computer, however, utilizes qubits that can take various states all at once, otherwise known as "superposition."

In particular, scientists have observed what is known as an "emergent" particle at the interface between two topological insulators. This particle is the Majorana fermion, a new class of matter that, unlike classical particles, possesses its own antiparticles (particles that annihilate their natural counterparts which possess the same mass but opposite electric or magnetic properties). These particles possess quantum numbers that are different from ordinary electrons are what could make topological insulators an ideal application in quantum computing.(6)

Quantum computers are great for solving optimisation problems from figuring out the best way to schedule flights at an airport to determining the best delivery routes for the post. Google recently had its quantum computer complete a problem that would have taken a classical computer 10000 years to complete and the google computer "Sycamore" did it in 200 seconds. This kind of efficiency is what makes quantum computers hugely important and therefore makes the implementation topological insulators pivotal.

3.2. **Energy Efficiency.** People spend a huge amount of time on the internet whether it be on their phones using social media apps or on their computer working and studying. And the internet holds an unimaginable amount of information for us to tap into. But this information and the demand for it means we need to build servers to hold this information. Server farms and Data centers are the factories of the digital age. There may be no chimney with black smoke pouring out of it to tell you these centers are bad for the environment but the carbon footprint of these data centers is just as big. The biggest, covering a million square feet or more, consume as much power as a city of a million people. In total, they eat up more than 2 percent of the world's electricity and emit roughly as much CO2 as the airline industry. If the global IT industry were a country, only China and the United States would contribute more to climate change. And with global data traffic more than doubling every four years, they are growing fast. (7)

Topological Insulators could be the key to making the internet green and reducing its carbon footprint. Resistance in the types of material like silicon lead to the build up of heat which is why sometimes your laptop or phone can feel hot. Most of the energy used in data centers is used to keep processors cool because of the

resistance of silicon transistors. Whereas topological insulators have little or no electrical Resistance and therefore will have less build up of heat and not need as much cooling. Topological insulators are 1,000 times more energy-efficient that the conductors we use in technology today and implementing this technology is the next step to the next technological age.(9)

4. Conclusion

Topological Insulators are a new and unique type of material with superconductive properties and applications in almost any electrical device. Their insulating interior and superconductive surface separates them from ordinary conductors whose entire mass has electrical flow. The absence of electrical resistance in topological insulators makes them a greener alternative to the materials currently in place in electronics as they don't require huge amounts of energy to cool.

As the technology industry continues to grow exponentially and with it, the demand for server farms and data centers, it's reassuring to know that we can have the potential to make these centers have less impact on the environment and more efficient at the same time by using topological insulators. We can also expect the use of topological insulators to have a great impact in the race for Quantum Supremacy (the race for the first commercially viable quantum computers) as it reaches it's conclusion.

The field of topological insulators is still at an early stage in development since we discovered the basic properties in 2007 and there is still a lot to be done to utilize their full potential. They are the result of interplay between theoretical insight and experimental discoveries and we can be optimistic that these materials will continue to develop in exciting new directions.

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PERSISTENT HOMOLOGY: HOW IT IS USED TO ANALYSE POINT CLOUD DATA SETS

MATT O'REILLY AND PAUL ARMSTRONG

Abstract. Topological data analysis (TDA) is a way to analyse datasets using topological techniques. Extracting information from datasets that are high-dimensional and noisy can be challenging. The main tool that is used in TDA is Persistent Homology, It is a method for computing topological features of a space at differential spatial resolutions. Simply put Persistent homology is an algebraic tool for measuring topological features of shapes and functions.

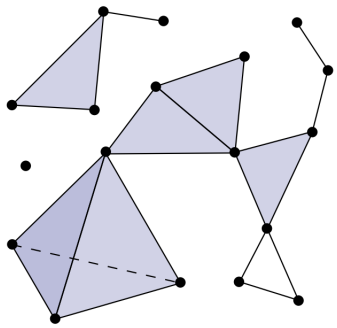
1. What is Homology?

Homology in general is a way to associate other parts of abstract algebra with topological spaces,this is typically done by using algebraic concepts such as open balls to describe the simplicial complexes (which we will define below) of a topological space. Homology is a sub section of Topology which slowly developed with the subject. Homology has been used as a means of categorising n dimensional manifolds alongside the euler characteristic. After Euler, Reimann defined genus and n fold connectedness as invariants of a topological space. Later Mark Goresky and Robert MacPherson would define persistent intersection homology (the basis of this paper) and prove it to be another topological invariant. [2]

2. Background

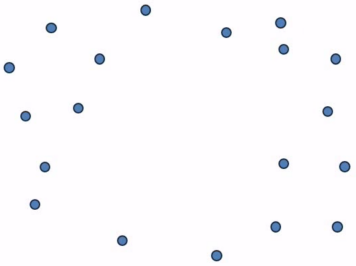
Persistent homology is an algebraic method for measuring topological features of shapes and functions. Persistent features of data are deemed more likely to represent true features of the underlying space than noise. To begin talking about topological features of a given data set we must first introduce the concept of a Simplicial Complex. A **simplicial complex** is the combination of a number of simplices.

2010 *Mathematics Subject Classification.* 55N33.



3. Topological Features

Example: What topological features does the following data exhibit?



The data appears to be represented by a ring. However, this is just intuition. How can we verify that is a topological feature? Let X represent the set of data points.

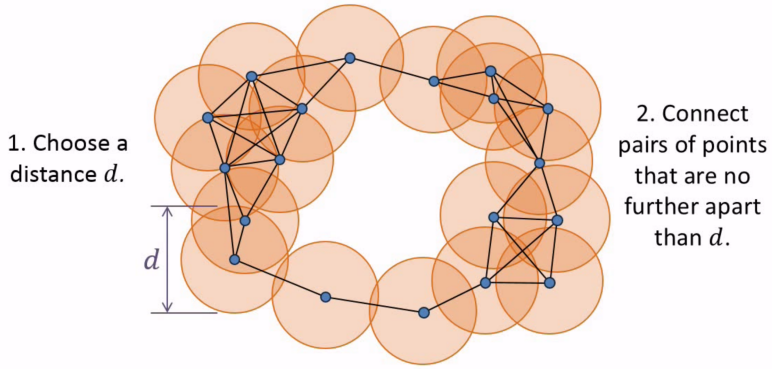
3.1. How do we turn our data points into shape?

Fix a real-valued distance, $d > 0$. Take the open ball of radius d around every point (x,y) in the data-set.

$$\forall\, x, y \in X,\ \exists\, d > 0\ B_d(x, y) = \{(a, b) : |(x, y) - (a, b)| < d\}$$

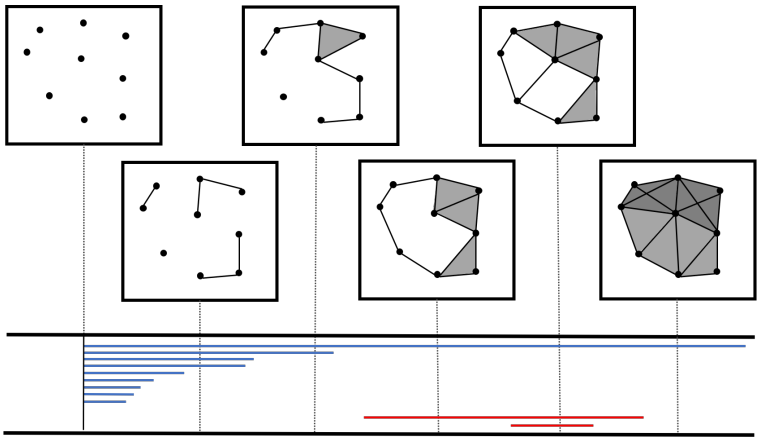
If n+1 circles intersect, draw a n-simplex between the n+1 points at the centers of the circles.

This gives a simplicial complex which roughly describes the shape of the data



3.2. **Choosing the right distance.** Our final hurdle is choosing the right d, If we choose a d too small then we might detect multiple connected components. This we can call noise. If we choose d too large then we get a giant simplex of all of the connected points. This has the trivial topology. For different values of d the topological structure will change.

So, What distance d should we choose? A solution is to consider all values of d. We let d vary over a range of values and count the components at each value of d, We record a barcode of the data which is seen below. [3]



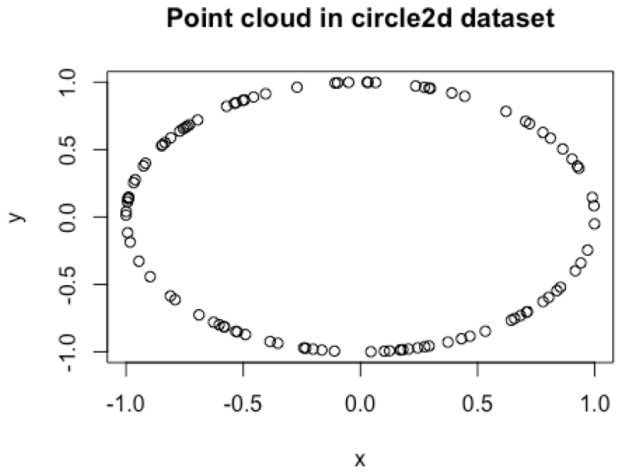
The small holes which we record are due to irregularities and noise, These are represented by the short bars in the barcode. The large bars represent significant features of the data

4. Topological Data Analysis using R Studio

```
library("TDAstats")

head(circle2d)
##              [,1]       [,2]
## [1,] -0.09728967  0.9952561
## [2,] -0.69421384  0.7197688
## [3,] -0.89704859 -0.4419319
## [4,]  0.83824069 -0.5453004
## [5,]  0.29894927  0.9542690
## [6,]  0.80302853 -0.5959406

plot(circle2d, xlab = "x", ylab = "y")
```



Calculating persistent homology

```
circle.phom <- calculate_homology(circle2d)
head(circle.phom)
##      dimension  birth        death
## [1,]         0      0  0.0007823978
## [2,]         0      0  0.0024476588
## [3,]         0      0  0.0043431663
## [4,]         0      0  0.0050472712
## [5,]         0      0  0.0055217817
## [6,]         0      0  0.0058158724
tail(circle.phom)
##       dimension      birth      death
##  [95,]         0  0.0000000  0.1655777
##  [96,]         0  0.0000000  0.1676424
##  [97,]         0  0.0000000  0.1724907
##  [98,]         0  0.0000000  0.1962725
##  [99,]         0  0.0000000  0.2085510
## [100,]         1  0.2232352  1.7339087
```

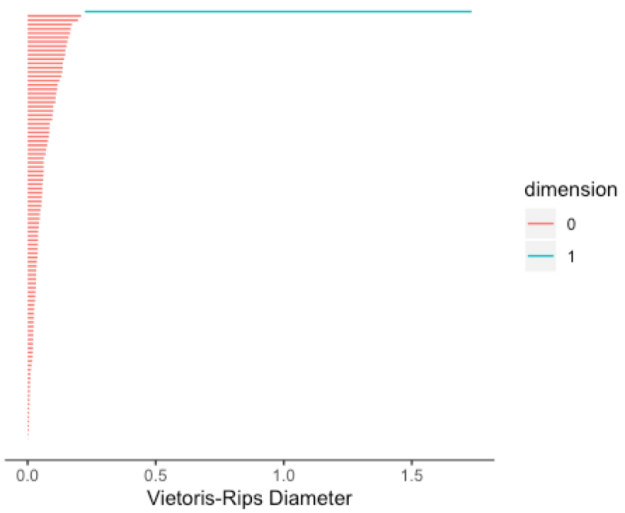
Each row in the homology matrix returned by the calculate homology function (variable named circle.phom) represents a single feature (cycle). The homology matrix has 3 columns in the following order:

1. Dimension: if 0, represents a 0-cycle; if 1, represents a 1-cycle; and so on.
2. Birth: the radius of the Vietoris-Rips complex at which this feature was first detected.
3. Death: the radius of the Vietoris-Rips complex at which this feature was last detected.

Persistence of a feature is generally defined as the length of the interval of the radius within which the feature exists. [1]

the vietoris rips complex is a special simplicial complex which for a defined metric and a value for distance e connects all vertices of distance less than e with an edge.

```
plot_barcode(circle.phom)
```



We can see a number of 0-cycles (The Red Bars) which represent noise in the data. The single 1-cycle at the top of the barcode (The Blue bar) is what is of interest to us. We can see that it clearly is a significant feature.

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SERGEI NOVIKOV, THE WORK WHICH EARNED HIS FIELDS MEDAL, AND HIS CONTRIBUTION TO TOPOLOGY.

DAVID O'BRIEN AND PAURIC MCSHANE

Abstract. In this project we will talk about the life's work of Sergei Novikov. We will discuss his contributions to the world of mathematics with particular focus on topology. This presentation will include details about the work that earned him his Fields medal and many other awards. We chose to study Novikov as we were interested in his work and thought some of the other students in topology would also be interested in some of the things he has accomplished.

1. General Information

Sergei Novikov is a Russian mathematician, born in Nizhny Novgorod, Russia on March 20th 1938. He graduated from Moscow State University in 1960. Novikov then went on to get his PhD and Doctor of Maths by 1965 from the V.A. Steklov Institute of Mathematics in Moscow. some of his achievements include winning the Fields Medal in 1970 for his contributions to the field of topology, he also won the Wolf Prize in 2005. He has published a wide variety of papers all based around the subject of topology from The Topology of Foliations in 1965 to Singular Finite-gap Operators and Indefinite Metrics in 2009.

2. Early Work

After graduating in 1960 Novikov produced many papers on the subject of Topology which ultimately led to the winning of his Fields Medal in 1970. Probably his most notable paper from this era was his paper on the Topology of Foliations. A foliation is an equivalence relation on an n-manifold, the equivalence classes being connected, injectively immersed sub-manifolds, all of the same dimension p, modeled on the decomposition of the real coordinate space R^n into the cosets $x + R^p$ of the standardly embedded subspace R^p[5].

The aim of this paper is to deal with relation between the topological properties of the manifold and those of the leaves. A manifold is a topological space which locally resembles a real n-dimensional space. The leaves are the equivalence classes of the foliation. The simplest example of a foliation given on the paper is a fiber bundle with a circle as the base, the leaves being the fibers. In this case, the foliation is connected, and the semi-group $t(A)$ of closed transversals, for any leaf A, is monomorphically embedded in the group $\pi_1(M^n)$, and consists of all elements of $\pi_1(M^n)$ which project into a positive multiple of the base-circle. There are no

2010 *Mathematics Subject Classification.* Primary 22.

limit cycles, and the subgroups $\Pi_j^i(A)$ are trivial for all leaves. If the base is a segment, then each leaf is a connected component[4].

The main part of the paper and the most impressive part is Novikovs closed leaf theorem. This spans over 9 pages in the paper including a two page conclusion. The Theorem he eventually proves is, let $n = 3$, and suppose that for some leaf $A \subset M^3$ of a smooth orientable foliation on a compact manifold M^3 the group $\Pi_1^j(A)$ is nontrivial. Then A is compact and lies on the boundary of a connected component of the foliation[4]. This proof along with the paper meant that his work couldn't go unnoticed so he was awarded the Fields Medal a few years later.

3. The Novikov Conjecture

The Novikov conjecture is one of the most important and well-known unsolved problems in topology. It has to do with the question of the relationship of the characteristic classes of manifolds to the underlying bordism and homotopy theory[6]. The history of the Novikov Conjecture starts with the Hirzebruch signature theorem $[Hir]$, which expresses the signature of an oriented closed $4k$-dimensional manifold M in terms of characteristic classes:

$$Signature(M) = \langle L(M), [M] \rangle \in \mathbb{Z}[2]$$

Here the $L(M) \in H^4(M; Q)$ is the L-class of M, a certain formal power series in the Pontryagin classes $p(M) \in H^4(M)$ with rational coefficients. The formula is surprising in that the left hand side is an integer which only depends on the structure of the cohomology ring of M, whereas the right hand side is a sum of rational numbers which are defined in terms of the differentiable structure. The Pontryagin classes can be expressed through the Riemann curvature tensor of any Riemannian metric as differential forms whose integrals along the cycles yield a "*rational*" homological Pontryagin class; these integrals are integers which do not change under small variations of metric (i.e. are diffeomorphism invariant). They determine only the image of class pk in the groups $H^4k(M, Q)$ with rational coefficients.

This Conjecture earned Sergi Novikov his **Field's Medal.**

4. Fields Medal

The Fields Medal was created in 1936 when John Charles Fields had $ 2,700 left from the 1924 International Congress of Mathematicians in Toronto. The winners have generally made contributions that opened up whole fields or integrated technical ideas and tools from a wide variety of disciplines. The majority of winners worked in highly abstract and integrative fields such as algebraic geometry and algebraic topology. Novikov won his medal in 1970, alongside Heisuke Hironaka, John G. Thompson, and Alan Baker. The Fields medal is considered the Mathematicians equivalent of the Nobels Prize. [3]

5. Novikov's contribution to Topology

Along with his award-winning research on Foliations, Novikov was among fellow topologists that carried out important research on **Geometric Topology**, being pioneers in the *Surgery theory* method for classifying high-dimensional manifolds.

In 2005 Novikov was awarded the Wolf Prize for his contributions to algebraic topology, differential topology and to mathematical physics. He is one of just eleven mathematicians who received both the Fields Medal and the Wolf Prize[1]. It is

awarded in six fields: Agriculture, Chemistry, Mathematics, Medicine, and Physics. Until the establishment of the Abel Prize, the Wolf Prize was probably the closest equivalent of a "*Nobel Prize in Mathematics*", since the more prestigious Fields Medal was only awarded every four years to mathematicians under forty years old.

In the early 1970s Novikov turned his attention to mathematical physics, initially contributing to general relativity and conductivity of metals. He constructed a global version of Morse theory on manifolds and loop spaces that had novel applications to quantum field theory. His most significant achievements in mathematical physics flow from his introduction of algebraic-geometric methods to the study of completely integrable systems. These include a systematic study of finite-gap solutions of two-dimensional integrable systems, formulation of the equivalence of the classification of algebraic-geometric solutions of the KP equation with the conformal classification of Riemann surfaces, and work on "almost commuting" operators that appear in string theory and matrix models.

6. Sergei's Later work

In 1982, Novikov was also appointed the Head of the Chair in Higher Geometry and Topology at the Moscow State University. As of 2004, Novikov is the Head of the Department of geometry and topology at the Steklov Mathematical Institute. He is also a Distinguished University Professor for the Institute for Physical Science and Technology, which is part of the University of Maryland College of Computer, Mathematical, and Natural Sciences at University of Maryland, College Park and is a Principal Researcher of the Landau Institute for Theoretical Physics in Moscow.

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THE ORIGINS, EVOLUTION AND THEORY BEHIND KNOTS

MATTHEW BOURKE, DYLAN GALLAGHER, AND KEVIN MANNION

(Communicated by David Futer)

Abstract. In the paper we will explore the fundamentals of what a knot is.

1. Introduction

Knot theory is the study of mathematical knots. A mathematical knot differs somewhat to traditional knots we think of everyday, like the knot you tie with your shoe laces. Mathematical knots differ as their ends are connected and cannot be undone. The most simple knot one can imagine is that of a ring which is mathematically known as an 'unknot'. Mathematical knots exist on a 3-dimensional Euclidean space, R^3.

2. Origins of mathematical Knot Theory

Mathematical knot theory can be traced back as far as 1833, when Carl Fredrich Gauss developed the the Gauss Linking Integral. The Gauss Linking integral was used to compute the linking number of two separate knots. Progress on knot theory slowed until the early 20th century. In the early 20th century, famous topologists such as Kurt Reidemeister, J.W Alexander and Max Dehn explored and developed new techniques to deal with knots, such as the Reidemeister moves and Alexander polynomial.[**BritannicaWebsite**]

In the late 20th century, Knot theory was in decline until Wolfgang Haken discovered an algorithm that could determine whether or not a knot is non-trivial. Haken also discovered a strategy by which solving the general knot recognition was made simpler. In layman's terms, this means Haken's discovery could determine whether or not two given knots were equivalent.

2.1. **Knot theorist - Field Medal.** Vaughan Jones won a fields medal in 1990 for his work on knot polynomials. His studies lead to the solution of a number of classical problems of knot theory. On receiving the Fields medal in Kyoto, Jones wore a New Zealand All Blacks Jersey. [**Jones**]

3. Classifications of Knots

3.1. **Trivial Knot.** A trivial knot is best represented by an elastic band. From a certain projection, trivial knots may appear to have a number of crossings, but when we reduce the number of crossing to the minimum amount possible we will be left with a ring.

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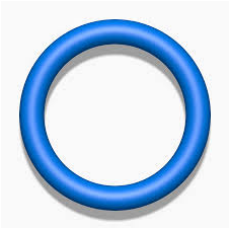
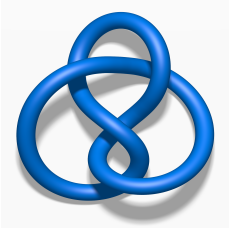


3.2. **Trefoil Knot.** A trefoil knot, also known as a threefoil or overhead knot, is the simplest example of a non-trivial knot. The trefoil knot got its name from the three-leaf clover (or trefoil) plant. The trefoil knot has three crossings. With continuous deformation we can produce more crossings however it is impossible to have fewer than three crossings. The trefoil knot is chiral (it is not identical to its mirror image). To distinguish between a trefoil and its mirror image we identify the left-handed trefoil and the right-handed trefoil. It is impossible to continuously deform from a left-handed trefoil to a right-handed trefoil, and conversely right to left.

3.3. **Figure 8 knot.** A Figure-eight knot is the only unique knot with a crossing number of four. The knot is also called the Listing's knot or a Cavandish Knot. Listing was a student of Gauss who furthered the study of knot theory after Gauss. The Figure-eight Knot has the third smallest possible crossing number. Unlike the Trefoil knot it is the same as its mirror image.



3.4. **Gordian knot.** For the likes of the trefoil and figure-eight knots we can easily determine the minimum number of crossings, but what happens when we encounter a very complicated knot with a high number of crossings? A Gordian Knot originates from a legend in which Alexander the Great cut a famously intricate knot which could not be unravelled. In mathematics a Gordian knot is seen as an exceedingly complicated problem, or deadlock. A modern topologist assumes that knots are constructed out of perfectly flexible, stretchable, infinitely thin string. Under those assumptions, if the Gordian knot were really a knotted loop, then it would not have been possible to untie it or to manipulate it so it will return to the form of a simple loop that does not cross itself.

4. Applications of Knot Theory

After a Scottish physicist Peter Tait experimented with smoke rings, Sir William Thomson presented an idea that atoms were knots of swirling vortices in the ether. Chemical elements would thus correspond to knots and links. Tait and Thomson both believed that understanding and classifying all knots would explain why atoms absorb and emit light at only discrete wavelengths. Unfortunately, the luminiferous ether was not detected by the famous Michelson-Morley experiment. Therefore vortex theory became completely obsolete and thus knot theory faded out of interest in physics.

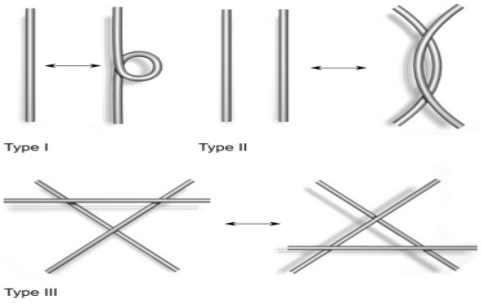
Nowadays knot theory is used in only a few disciplines. Knot theory is used by biologists in the study of both DNA and Protein. Topology, and specifically knot theory, facilitates the unknotting of DNA. DNA can be visualised as two long strands, intertwined millions of times, tied into knots and coiled excessively. When researchers apply knot theory to unravel the mysteries held inside the strands of DNA, they can neatly arrange the data making it easier to understand.

Knot theory can also be used in a variety of other technologies, including applications in cryptology, motion-planning in robotics and GPS. Knot theory also has it uses in the solving of certain puzzles. Disentanglement puzzles, which consist of two or more 'parts'that are entangled together and require the solver to separate, subject to several geometric constraints [**Horak**]

5. Methods to unknotting

5.1. **Reidemeister Moves.** Reidemeister moves proved that knots exist which are distinct from the unknot. This can be shown by reducing it to a sequence of three types by deformation. The moves include:

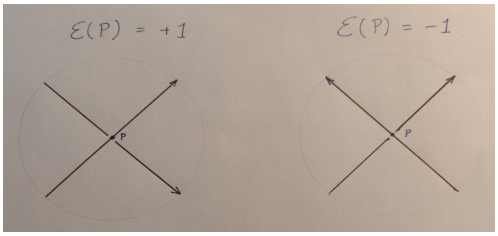
(i) The twist move
(ii) The poke move
(iii) The slide move



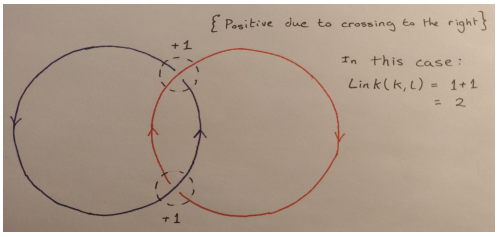
5.2. **The use of in-variants.**

(1) Minimum number of crossings in a diagram. A reduced alternating diagram exhibits the minimal number of crossings
(2) Gauss linking number: (The linking number of a link of two knots) Link in general is an intertwined combination of knots. Two knots linked between each other E(P) is the crossing which is either positive or negative. The link between two knots:

$$Link(k,l) = \sum_{n=1}^{\infty} E(P) \tag{5.1}$$



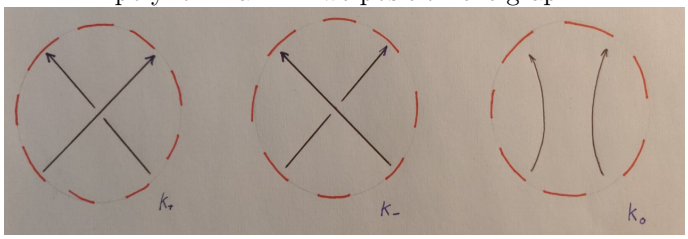
Example of Gauss linking number:



(3) Alexander Conway polynomial:

$$Knot - K \rightarrow \Delta(K) \tag{5.2}$$

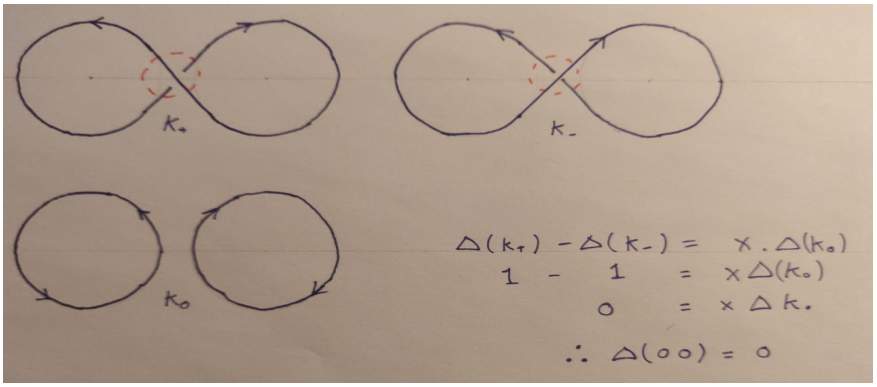
polynominal in x we position the graph



- $\Delta(0) = 1$ $(un - knot\ evaluation = 1)\Delta(K_+) - \Delta(K_-) \ = \ x.\Delta(K_0)$

- $\Delta(K_+) - \Delta(K_-) \ = \ x.\Delta(K_0)$

Example of Alexander-Conway polynomial:



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NOTABLE TOPOLOGISTS WHO HAVE BEEN RECIPIENTS OF THE FIELDS MEDAL.

EMILY MCDONAGH, LISA MURPHY AND SARAH EGAN

(Communicated by David Futer)

Abstract. In this paper we give an account of what a Fields Medal award is and what its significance is in Mathematics. After that we go on to analysis 3 different topologists and what contribution to the field led them to being awarded a fields medal. The three topologists that we chose were Vaughan Jones, Grigori Yakovlevich Perelman and René Frédéric Thom.

1. What is a fields medal

The Fields Medal, (officially known as International Medal for Outstanding Discoveries in Mathematics) is an awarded every four years at the International Congress of the International Mathematical union (or the IMU for short) to at least two, with a preference for four mathematicians under the age of 40. It is regarded as one of the greatest honours any one mathematician can receive in their career and is sometimes even referred to as the mathematician's Nobel Prize. The Fields Medal is awarded to recognise outstanding achievements as well as any promising future work that a mathematician may have.
The fields Medal originated through funds raised by a Canadian professor of mathematics named John Charles Fields and was first awarded in 1936 along with a cash prize of 15,000 US dollars. Since its commencement in 1936, 60 young mathematicians have been awarded fields Medals for their contributions in many different Fields on mathematics. Only one woman among the 60 mathematicians has ever been awarded the Fields medal and that was Maryam Mirzakhani in 2014.

The fields medal can be awarded in many aspects of mathematics including algebraic geometry, dynamic systems, analysis, differential equations, mathematical physics and many many more. For the purpose of this paper we will concentrate on fields medals awarded for contributions in topology and we have chosen 3 topologists that made major contributions to the field.

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2. Vaughan Jones

2.1. **Early Life and Career.** Vaughan Jones grew up in Auckland, New Zealand where he went to the University of Auckland. He graduated from the University of Auckland with first class honours in his bachelor and went on to do a master degree in science. After this and on receipt of a scholarship from the Swiss government, Jones decided to move to Geneva to continue his studies. On arrival in Geneva in 1934 Jones immediately joined the École de Physique to study physics however a year and a half later changed over to the École des Mathématiques. It was during there years and under the supervision of Swiss mathematician André Haefliger that Jones was awarded his Docteur és Mathématiques. This would be the first of many achievements in Jones career. (David salisbury, 2011)

2.2. **His work and fields medal.** Jones is most credited for his work that led to his awarding of the Fields medal which fun fact he accepted while wearing a New Zealand rugby jersey. He was awarded the medal in 1990 aged 37 for his discovery of the link he made between statistical mechanics and knot theory. Specifically, he a new mathematical expression that is now famously known as the Jones polynomial that distinguishes between different kinds of knots. His discovery is very vital and in fact has formed the basis for an entirely new branch of mathematics called quantum topology that allows physicists to complicated three-dimensional spaces filled with holes and loops.

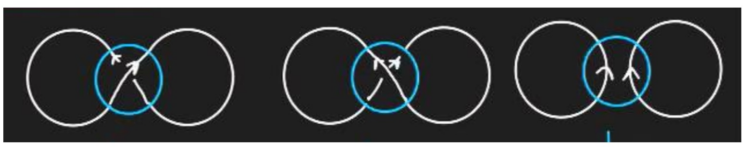
2.3. **Jones polynomial.** Jones polynomials are Laurent polynomials expressed in t assigned to a R^3 knot. The Jones polynomials are denoted $VL(t)$ for links and $VK(t)$ for knots and follow 3 specific properties:

(1) The Jones polynomial of any link (L) is contained in the ring $\mathbb{Z}[t^1/2, t^-1/2]$
(2) The jones polynomial on the un-knot $K(O) = 1$
(3) The Skein relation applies to the Jones polynomials and is used to calculate one of V(L+), V(L-) or V(L0) when two variables are known.
   (a) where the Skein relation is:

$$t^-1V(L+) - tV(L-) + (t^-1/2 - t^1/2)V(L_0) = 0$$

   (b) and where $L+, L-, L_0$ are oriented links with diagrams that are equal except in a small region where they differ as follows:

Above $L+$ is referred to as a positive crossing, $L-$ is referred to as a negative crossing and L_0 is referred to a no crossing. It should be noted that any two links are crossed in one of these 3 ways and can also be manipulated from one type to another. (Stephen Bigelow paper.)

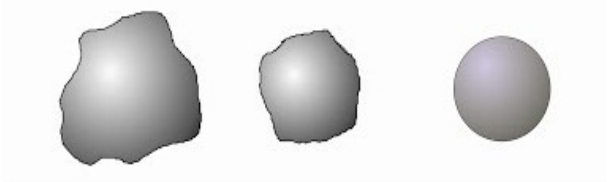


3. Grigori Yakovlevich Perelman

3.1. **Backround.** Grigori Yakovlevich Perelman was born on the 13th of June 1966. Born in Russia he has made many contributions to the field of topology through a number of proofs and was awarded the fields medal in 2006 for "his contributions to geometry and his revolutionary insights into the analytical and geometric structure of the Ricci flow". However, Perelman declined the medal. He was the first person in history to reject to award as he claimed he had no desire for money or fame. Perelman's proof of the Poincaré conjecture was recognised as the scientific "Breakthrough of the year" in the scientific journal in 2006.

3.2. **The Problem.** Henri Poincaré put forward the conjecture in 1904. He asked the Question "If a compact three-dimensional manifold M3 has the property that every simple closed curve within the manifold can be deformed continuously to a point, does it follow that M3 is homoeomorphic to the sphere S3?" The Poincaré conjecture is a theorem about the characterization of the 3-sphere, which is the hypersphere that bounds the unit ball in four dimensional space. The conjecture says "every simple connected, closed 3-manifold is homoeomorphic to the 3-sphere." Inspiring topologists to try prove what became known as The Poincaré conjecture, it is one of the oldest and simplest stated problems of topology. It became one of the seven millennium prize problems and is the only solved one.

3.3. **Perelman's proof.** Perelman's proof was achieved through the use of the dynamics of heated objects. Topologists typically used techniques of cutting and pasting in their attempts to prove the problem. The idea that as objects get heated they become rounder and rounder. This technique is known as the Ricci flow which was defined by Richard S. Hamilton. (Lê Nguyên Hoang, 2013- Perelmas article) Eventually they will form spheres. By using this idea to deform into one of the fundamental classes of closed 3-manifolds. If the initial 3-manifold is simply connected, it gets deformed into 3-sphere, proving the Poincaré conjecture. How can one be sure that infinitely many cuts weren't necessary? He proved this cannot happen using minimal surfaces on the manifold. Hamilton showed that the area of a minimal surface decreases as the manifold undergoes Ricci flow. Perelman proved that eventually the area is so small that any cut after the area is that small can only be chopping off three-dimensional spheres and not more complicated pieces. Below you can see an illustration of **The Ricci Flow being used to warp manifolds**



3.4. **Aftermath.** According to Michael William's presentation on Perelman, his proof had a large impact in both mathematics and physics. Some topologists believe Perelman's proof will be an important tool for defining the universe itself. Unfortunately, Perelman has cut ties with the mathematically community after the disappointment he felt in it. Not only did someone try claim his work on the Poincaré Conjecture, the one person he admired and wished to work with ignored him both before and after his proof was made. It is unknown if he still practices mathematics.

4. René Frédéric Thom

4.1. **About.** René Frédéric Thom was a French mathematician. Thom's career was one of huge success, spanning a number of different topics, he contributed many theorems, concepts and findings to areas including differential topology, singularity theory, cobordism theory and many more. (Wikipedia (1)) In 1958 at the International Congress of Mathematicians in Edinburgh, he received the Fields medal for his work in comprising the foundations of cobordism theory. As a young mathematician he was recognised for substantial work in this area.

4.2. **Problem.** Cobordism Theory is an area of mathematics that studies manifolds modulo cobordism relation. The underlying concept of this theory is that if the boundary of the union of two disjoint manifolds is a different manifold then the two original manifolds are considered to be the same (Vito Cruz). Poincaré was the first academic to consider the notion of cobordism and then Pontrjagin after that. It was not until Thom established a more concise approach and utilised it in the computations of the unoriented cobordism groups (Vito Cruz). Thom studied these in two steps, the first showed that cobordism groups were related to homotopy groups of a certain Thom space. It is the development of the foundational concept of a Thom space that is René Frédéric Thom's most recognised works.

4.3. **Proof.** A Thom space is a topological space associated with a vector or sphere bundle over any para-compact space (a topological space where every open cover has an open refinement that is locally finite) (Wikipedia). Its role is to enable mathematicians to reduce geometric problems to hompotopic topology problems and then hence to reduce them further to algebraic problems. Thom spaces are constructed as follows.

- Let $\rho : E \to B$ be a vector bundle rank n, with paramount space B.
- For each x ϵ B, $\exists$ a fiber E_x that is known as a vector space in n-dimensions.
- A sphere bundle, also of n-dimensions can be formed one way by finding the Alexandroff compactification of each E_x and combining them to form $Sph(E) \to B$.
- To obtain the Thom space T(E) from Sph(E) we need to find all the new points to a single point: . This enables us to find T(E) as the quotient of Sph(E) by B . The basepoint of T(E) is the combination of these single points. (Vito Cruz)

We can conclude that if B is compact then T(E) is the Alexandroff compactification of E.

Note: Fiber: $If X \to Y$, the fibre of element y is the inverse image of a singleton y under f.

4.4. **Aftermath.** From this realisation came the Thom isomorphism which was proved by Thom in his famous 1952 thesis, which contained his considerable foundational work in cobordism.

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Poincare Conjecture

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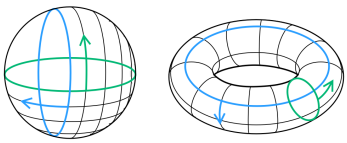
1 Abstract

The topological topic that we decided to choose that wasn't covered in the MA342 exam syllabus is Poincare Conjecture. Poincare's Conjecture states that "every simply connected closed 3-manifold is homeomorphic to the 3-sphere."



2 History

Henri Poincare posed the question, could we understand the shape of the universe that we live in by using mathematics. For example, is the universe like the inside of a sphere where you can move around freely or is it like the inside

of a big inner tube where you keep going around and around or something more complicated like a pretzel shape[6].

His original conjecture stated, "consider a compact 3-dimensional manifold V without boundary. Is it possible that the fundamental group of V could be trivial, even though V is not homeomorphic to the 3-dimensional sphere?". This can be rewritten as "every loop can be shrunk to a point."

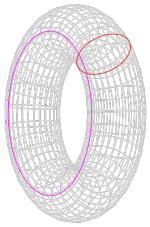
The Poincaré conjecture later became known in its standard form "every simply connected, closed 3-manifold is homeomorphic to the 3-sphere." This problem became one of the most important unsolved problems in algebraic topology.

3 Attempted Solutions

Many proofs were attempted since its conception. In the 1930's J. H. C. Whitehead claimed he had a proof but later retracted it. Many more followed but also failed after discovering flaws in their proofs.

In 1958, R. H. Bing, an American mathematician proved a weak version of the Poincaré conjecture. He found that if every simple closed curve of a compact 3-manifold is contained in a 3-ball, then the manifold is homoeomorphic to the 3-sphere [2]. In 1978, Włodzimierz Jakobsche added to this and showed that if Bing's conjecture was true in the third dimension, then the Poincaré conjecture must also be true.

The conjecture was especially difficult to prove as small errors in the proofs were very difficult to detect. Mathematicians began approaching alleged proofs with scepticism. It was as a thorny problem to solve [1]. It became known as one of the most important open questions in topology. Because of this it was included in a list of the seven Millennium Prize Problems and anyone who could find a solution to the conjecture would be awarded a 1 million dollar prize by the Clay Mathematics Institute.



4 Dimensions

An American mathematician named Stephen Smale proved the generalize Poincare Conjecture for all dimensions greater than or equal to five. For this proof he won a Fields Medal in 1966.

Another American Mathematician by the name of Michael Freedman won a Fields Medal for proving the generalized Poincare Conjecture for dimensions equal to four. However, nobody had been able to prove or disprove Poincare's Conjecture.

The conjecture was still not proven in dimension 3. Thurston's work on the geometrization conjecture led the way by governing the framework of 3-manifolds [3]. Which ultimately led to mathematicians general acceptance that the conjecture was true.

5 History of the Proof

Richard S. Hamilton introduced the Ricci flow on a manifold in a 1982 paper. He used this to prove specific cases of the Poincaré conjecture. However, he was never able to prove it entirely. He also laid out a program for proving Thurston's geometrization conjectures.

Almost a century after the conjecture was created, in 2002 a Russian mathematician by the name Grigori Perelman was able to complete Hamilton's program and came up with a working solution. Perelman published his work in three papers where he proved the Poincaré conjecture along with the stronger general conjecture, Thurston's geometrization conjecture, completing the Hamilton program [4].

It took until 2006 for the proof to be verified. Several groups got together to complete the details of Perelman's proof. They found only minor omissions.

In October 2006 Grigori Perelman was awarded the Fields Medal by The International Congress of Mathematicians along with a 1 million dollar prize [5]. He refused to accept the award saying "I'm not interested in money or fame; I don't want to be on display like an animal in a zoo."

To mathematicians, Grigori Perelman's proof of the Poincaré conjecture was honoured as the Breakthrough of the Decade [5].

6 Poincare Conjecture

The Poincare Conjecture states, "every simply connected closed 3-manifold is homoemorphic to the 3-sphere." More colloquially, the conjecture says that the three-sphere is the only type of bounded three-dimensional space possible that

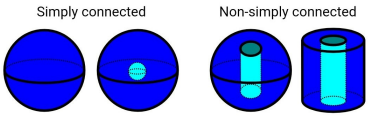
contains no holes. Breaking down the conjecture to understand it better poses three questions. What is a 3-sphere? What it means for two objects to be homeomorphic and what is a manifold?

For something to be defined as a 3-sphere it has to have three requirements. It must have a three-dimensional surface; it must sit in a four dimensional space and every point must be the same distance from the centre[7].

To prove that two objects are homeomorphic we must have a function that maps all of the first object onto all of the second object i.e. f(x)=2x. The inverse of f must map all of the second object onto all of the first object i.e. g(x)=x/2. Both functions must also be continuous. In Lehman's terms the function is continuous if you can draw it without lifting the pen.

A manifold informally is defined as a surface that locally resembles a tangent plane near every point. Examples of manifolds include a 2-sphere, torus and the function z=$y^2 + x^2$.

A path wise-connected domain is said to be simply connected (also called 1-connected) if any simple closed curve can be shrunk to a point continuously in the set. If the domain is connected but not simply, it is said to be multiply connected. Thus, if a space is simply connected, then it is connected[8].



So work on the Poincare conjecture has led to a deeper understanding of 3-manifolds in general. Thanks to Perelman's work he also proved the spherical space form conjecture which was conjected by Heinz Hopf in 1926 and Thurston elliptization conjecture which was conjected by William Thurston in 1980. The fundamental idea behind algebraic topology is to translate topological problems into algebraic problems.

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QUOTIENT SPACES

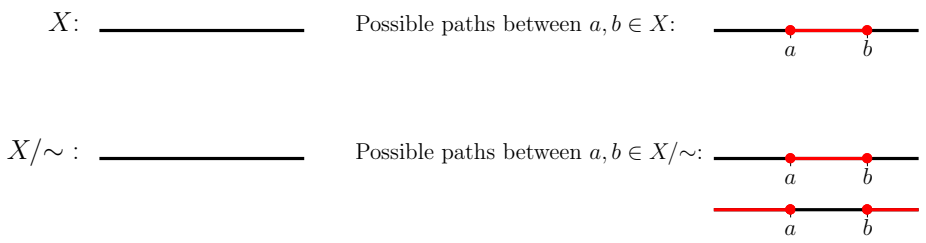
D. KIRBY, C. NOLAN, AND T. ROSENBERG

(Communicated by Shelly Harvey)

Abstract. A common way to describe topological spaces is to begin with one space and *identify* points with each other. For example, we might begin with a unit square and identify two edges i.e. consider them the same edge. When we do something like this we are really using the idea of an *identification space* or, as it's more commonly known, a *quotient space*. In this article, we hope to provide a formal introduction to this idea.

Let X be a topological space. We consider an equivalence relation $\sim$ on X and let $X/\sim$ denote the set of equivalence classes for this relation. $[x]$ denotes the equivalence class for some point $x \in X$. This is how we identify points with each other - we say they are equivalent.

Example 1: Let $X = [0, 1]$. Let $0 \sim 1$ and let $x \sim x$ for $x \in (0, 1)$. So the equivalence classes consist of $\{0,1\}$ and $\{x\}$ where $x \in (0, 1)$. We have 'joined' one end of the unit interval to the other. Let $a, b \in X$ where $a < b$. Before joining the ends, the only way to get from a to b was via the line segment $[a, b]$. After joining the ends, we can still go this way, but now we have an alternative route, as can be seen in the diagram below.



In terms of paths, at least, this new space should seem a bit like S^1, the unit circle.

So, now we know how to identify equivalent points on a set X, using $X/\sim$, but X is a topological space and as such has open sets defined on it. So what should the topology (open sets) defined on $X/\sim$ be? Well, there is a well-defined topology for this situation called the *quotient topology*. We won't explain why we choose this topology initially. Instead, we'll define it, then see how no other topology would be suitable for our purposes.

Definition The map $\pi : X \to X/\sim$ that maps $x \in X$ to the (unique) element of

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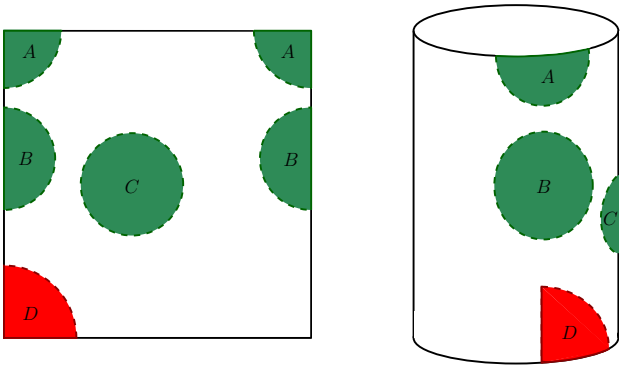
$X/\sim$ containing x is called the *canonical projection map* π. So for all $x \in X$,

$$\pi(x) = [x].$$

We define a topology on $X/\sim$ as follows:
Definition A subset $\tilde{O} \subseteq X/\sim$ is open iff $\pi^{-1}(\tilde{O})$ is open in X. It is not difficult to show that this actually defines a topology, see [3]. This is known as the *quotient topology*. Together with $X/\sim$ this is known as the *quotient space*.
Equivalently, the quotient topology is the largest topology on $X/\sim$ for which π is continuous.

Though we still haven't said why we choose this topology in particular, we now look at an example to show that some (but not all) of our open sets remain from before an identification was made.

Example 2: We consider the space $[0,1] \times [0,1]$ (the unit rectangle) where two edges are identified with each other. Specifically, for $(x_1, y_1), (x_2, y_2) \in (0,1) \times [0,1]$, $x_1 \sim x_2$ if $(x_1, y_1) = (x_2, y_2)$ and for $(x_1, y_1), (x_2, y_2) \in \{0,1\} \times [0,1]$, $x_1 \sim x_2$ if $y_1 = y_2$.



$[0,1] \times [0,1]$ $\quad$ $([0,1] \times [0,1])/\sim$

A, B, C and D would all be open sets in the usual product topology on $[0,1] \times [0,1]$ (or the subspace topology induced from $\mathbb{R}^2$; they are the same). However, in the quotient topology, D is no longer an open since its preimage from π is the sector of the red circle in the bottom left corner union a short line-strip on the right edge, which is clearly not an open set in $[0,1] \times [0,1]$ with the usual product topology.

We now look at some theorems which finally tell us why we should use the quotient topology when identifying points in a topological space. They also give us a way to prove that some quotient spaces are homeomorphic to other known spaces.

Theorem 0.1 (Characteristic Property of the Quotient Topoology)**.** *Let $X/\sim$ be a quotient space and let Z be an arbitrary topological space. A function $f : X/\sim \to Z$ is continuous iff $f \circ \pi : X \to Z$ is continuous.*

Proof. Let U be an open subset of Z. $f^{-1}(U)$ is open in $X/\sim$ iff $\pi^{-1}(f^{-1}(U))$ is open in X i.e. iff $f \circ \pi$ is continuous. □

Theorem 0.2 (Uniqueness of the Quotient Topology)**.** *Let (X, τ) be a topological space and $X/\sim$ the set of classes of an equivalence relation on X. The quotient*

topology on $X/\sim$, τ_q, is the only topology which gives $X/\sim$ the characteristic property.

Proof. Note: [2] provides useful diagrams of the following, if desired. We know that $(X/\sim, \tau)$ has the characteristic property described in 0.1. Suppose some other topology on $X/\sim$, τ_d, say, also gives it characteristic property. Let

$$\pi : (X, \tau) \to (X/\sim, \tau_q)$$

be the canonical projection map as described before. Let

$$\pi_d : (X, \tau) \to (X/\sim, \tau_d)$$

be map which takes points in X to the same points that π takes them to except now the domain $X/\sim$ has a different topology on it. $\pi(x) = \pi_d(x)\ \forall x \in X$, so if we didn't care about the topology of sets, π and π_d would be the same function. Let

$$1_{dq} : (X/\sim, \tau_d) \to (X/\sim, \tau_q)$$

be an identity map and let 1_{qd} and 1_{dd} be similarly defined (an identity map maps each element to itself, however in this case we consider different topologies in the domain and codomain). Clearly, $\pi = 1_{dq} \circ \pi_d$ and π is continuous. So 1_{dq} is continuous by the characteristic property of $(X/\sim, \tau_d)$ (1_{dq} is playing the role of f in 0.1). Similarly, $\pi_d = 1_{qd} \circ \pi$ so if π_d is continuous, 1_{qd} is continuous. To show that π_d is continuous, consider $\pi_d = 1_{dd} \circ \pi_d$. 1_{dd} is clearly continuous, so by the characteristic property of $(X/\sim, \tau_d)$, $1_{dd} \circ \pi_d$ which equals π_d is continuous (1_{dd} now playing the role of f in 0.1). So 1_{dq} and 1_{qd} are continuous, and if you think about what these maps do it's clear that $\tau_q = \tau_d$. □

Now we can start to see why we are using the quotient topology. It's the only one which has the characteristic property. You might ask why we care about our spaces having this property. Well, the next three theorems rely on it and they are extremely useful. We just need one more tool.

Definition A map from two topological spaces X and Y, $f : X \to Y$ is a *quotient map* if:

- f is surjective and
- a subset of Y is open iff $f^{-1}(Y)$ is open in X.

We use a quotient map $f : X \to Y$ to define an equivalence relation $X/\sim$ on X that is, elements which get mapped to the same point in Y are considered equivalent ($x_1 \sim x_2$ iff $f(x_1) = f(x_2)$).

Theorem 0.3. *If $f : X \to Y$ is a quotient map, then $X/\sim$ (as defined in the previous sentence), with the quotient topology, is homeomorphic to Y.*

Proof. $X/\sim$ consists of subsets of X, which are $f^{-1}(y)$, $y \in Y$. Define $h : X/\sim \to Y$ by $h(f^{-1}(y) = y$. Then h is a bijection and satisfies $h \circ \pi = f$ and $h^{-1} \circ f = \pi$. By 0.1, h and h^{-1} are continuous. Therefore h is a homeomorphism. □

This last theorem is extremely useful since it allows us to prove that one of our quotient spaces is homeomorphic to some known space (and all we have to do if find a surjective, continuous map which sends equivalent points to the same point. We now present a result about quotient maps which makes finding these maps a little easier.

Lemma 0.4 (Closed Map Lemma)**.** *Let X and Y be topological spaces and $f : X \to Y$ be a continuous, surjective map. If X is compact and Y is Hausdorff, then f is a quotient map.*

Proof. See **Theorem 4.3** and **Corollary 4.4** in [1]. It is a very procedural proof.

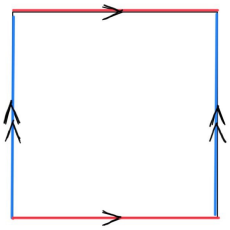
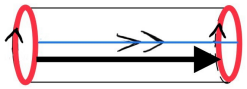
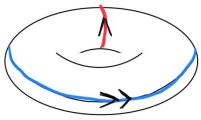
Now let's use our new theorems to show that the space we created in Example 1 is homeomorphic to S^1.

$$f : [0,1] \to S^1$$
$$x \mapsto e^{2\pi i x}$$

is a surjective map, $[0,1]$ is compact and S^1 is Hausdorff. So, by 0.4, f is a quotient map. f maps 0 and 1 to to the same point so f gives rise to $[0,1]/\sim$ (as described above 0.3. By 0.3, $[0,1]/\sim$ is homeomorphic to S^1.

Example 3: (from [3]) We use the concept of gluing (identifying) to tranform the unit square into a torus. We define an equivalence relation on the unit square by identifying boundary points. For each $y \in [0,1]$, we identify each $(0,y)$ boundary point to $(1,y)$; as a result, the unit square becomes a cylinder. In addition, for any $x \in [0,1]$ we identify each $(0,x)$ boundary point to $(1,x)$. The equivalence classes are given by:

$$\begin{aligned}
&[(x,y)] = \{(x,y)\}, && \text{for } 0 < x, y < 1,\\
&[(x,0)] = \{(x,0),(x,1)\}, && \text{for } 0 < x < 1,\\
&[(0,y)] = \{(0,y),(1,y)\}, && \text{for } 0 < y < 1,\\
&[(0,0)] = \{(0,0),(0,1),(1,0),(1,1)\}.
\end{aligned}$$

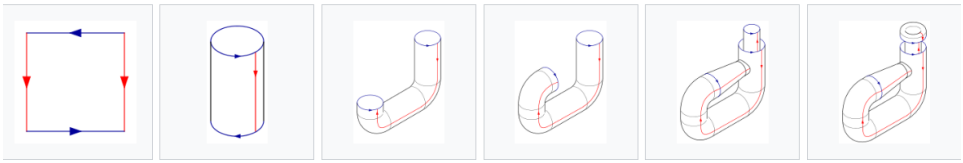
With these two equivalence relations defined, the unit square becomes a torus as the edges get identified and the four corners get identified. An equivalent way to describe the torus is the product space of two circles, $S^1 \times S^1$. Similarly to Example 1, to show that these spaces are homeomorphic, consider the map

$$f : [0,1] \times [0,1] \to S^1 \times S^1,$$
$$(x,y) \to (e^{2\pi i x}, e^{2\pi i y}).$$

By 0.4, f is a quotient map and f gives rise to the exact same equivalence classes on $[0,1] \times [0,1]$ as described above, so by 0.3, both spaces are homeomorphic.

Example 4: (Unit Square as a fundamental polygon of the Klein Bottle, from [4]). The Klein Bottle, it can be thought of as the quotient space defined by the unit square $[0,1] \times [1,0]$ wherein the relations which identify the sides $(0,y)$ and $(x,0)$ are $(0,y) \sim (1,y)$, for $0 \le y \le 1$, and $(x,0) \sim (1-x,1)$, for $0 \le x \le 1$.

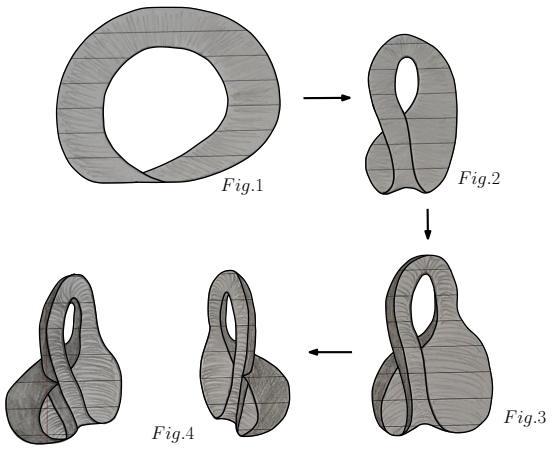
This can be visualised quite easily with help from the diagrams below:



https://commons.wikimedia.org/wiki/File:Klein_Bottle_Folding_6.svg (Public Domain)

The process observed in the above diagrams show a unit square with vertical edges oriented in the same direction, and horizontal edges oriented in the opposite directions. In order to 'glue' together the corresponding coloured edges with matching orientations, the vertical edges are 'glued' together first, creating a cylinder. In order to 'glue' the horizontal edges together, we must match their orientation. To do that, one of these edges (now one of the circular ends of the cylinder) is passed through the side of the shape, and out the top. It can then be folded down onto the other horizontal edge, to form a complete shape in the form of a Klein Bottle.

Example 5: (The Klein bottle as 2 Mobius Strips 'glued' together, from [5]). Another way to describe a Klein bottle using 'gluing' is to view it as the union of 2 copies of a Mobius Strip joined along their boundaries to form a closed shape. The described 'gluing' can be visualised through the diagram below, which shows the Mobius strips shaped to form a more obvious Klein Bottle shape, while still obviously remaining valid Mobius Strips:



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THE APPLICATION OF TOPOLOGICAL DATA ANALYSIS TO THE DIAGNOSIS AND CLASSIFICATION OF GBM TUMOURS

MARK DERVAN, RONAN LAVELLE-PRICE, AND NAOISE O'CALLAGHAN

(Communicated by American Mathematical Society)

Abstract. Glioblastoma multiforme, or GBM for short, is the most aggressive form of cancer that begins in the brain. The typical duration of survival of survival following diagnosis is 12-15 months, with fewer than 7% of people living beyond 5 years. Its rapid progression and the difficulty in performing a biopsy on brain tissue make images an important resource in the diagnosis of GBM. This paper will examine the mathematics behind application of topological data analysis to the diagnosis and classification of GBM tumours. To do this we will introduce the smooth Euler characteristic curve (SECT). Before introducing the SECT we will introduce simplicial complexes, persistent homology and the persistent homology transform(PHT).

1. Simplicial complexes

A simplicial complex is a set composed of points, line segments, triangles and their n-dimensional counterparts.
Simplicial complex $k \in \mathbb{R}$ is a collection of simplices in $\mathbb{R}^n$ such that:
Every face of a simplex in K is also in K and If any two k-simplices (α_1,α_2) are in k, then their intersection is either empty or a face of both (α_1,α_2)

2. Homology

Homology groups is an algebraic structure used to study holes in a topological space, in particular homology is concerned with studying the boundaries on holes. Homology is particularly relevant to this application as not only does it describe contrasting physical tumours, but it also captures some information about the stage of disease progression. In general the he 0th-degree homology captures the number of connected components in the shape, the 1st-degree homology captures the number of loops, and the 2nd-degree homology captures the number of voids. For example, Necrosis is a form of cell injury which results in the premature death of cells and multifocality is a radiological observation where individual tumour cells separate from the main masses and disperse elsewhere in the brain. From an imaging perspective, necrotic regions show up as dark regions (or holes) within a tumour, while multifocal tumours appear as segregated masses. It has been suggested that the more necrosis or multifocality there is in a GBM tumour, the more aggressive the disease. Applying homology to radiomic studies not only identifies such phenomena, but also tracks the number of times they occur and thereby provides a

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notional measure of disease severity. We will use the notation $\mathrm{H}_k(\mathrm{k})$ to denote the k-th homology group for the simplicial complex K. The image below shows necrotic regions in a brain scan (dark region surrounded by the whites region)

3. Persistent Homology

Persistent homology continuously tracks the evolution of homology at different scales. To do this we use the index, s, of a filtration. A filtration is a collection of simplicial complexes (k_s) where the index s induces totally ordered sets $\mathrm{k}_i \subseteq \mathrm{k}_j \forall \mathrm{i} < \mathrm{j}$. The shape information at each scale s is encoded by the homology groups $\mathrm{H}_k(\mathrm{k}_s)$ of the simplicial complex k_s . Computing persistent homology produces a collection of intervals for each degree of homology, where each interval indicates a k-dimensional topological feature (e.g. a connected component, loop, or void for a general, three-dimensional shape) that is "born" at the parameter value given by the left endpoint of the interval, and "dies" at the value at the right endpoint. The length of the interval corresponds to how long the topological feature "lives". These intervals are then plotted on a persistence diagram, where the start and end points are represented as a pair of points plotted on a plane with the x-value corresponding to the start time and y-value corresponding to the death time.

4. Persistent Homology Transform (PHT)

The PHT collects persistence diagrams of all degrees of homology in all possible directions and uses them to display shape information. The PHT is injective for 2 and 3-dimensioanl shapes between the shape itself and the infinite collection of persistence diagrams.

5. Smooth Euler Characteristic Transform (SECT)

Although the SECT uses the same underlying mathematical concepts as the PHT as opposed to producing a collection of persistence diagrams it instead produces a set of continuous pairwise linear functions. To understand how this is done we must first define the Euler Characteristic (EC).
In terms of homology the EC counts the Betti numbers, $\widehat{\beta}_k$, for the k-th homology group H_k, thus it reduces the mathematical description of holes in a topological space from an algebraic group structure to an integer

Let X be an arbitrary topological space, $H_k(X)$ be the k-th homology group of X, and β_k be the rank of $H_k(X)$. The Euler characteristic (EC) $\chi(X)$ of X is the alternating sum.
$\chi(X)=\beta_0-\beta_1+\beta_2-...=\Sigma(-1)^k(\beta_k)$
For a discretized shape or surface in three dimensions represented as a simplicial complex K, the EC may be analogously defined by the number of simplices in K by
$\chi(K)=V-E+F$
where V , E, and F are the numbers of vertices (0-simplices), edges (1-simplices), and faces (2-simplices), respectively. Just as homology may be augmented to persistent homology by considering a filtration, ECs may also be calculated with respect to a filtration. The result is an EC curve, which tracks the progression of the EC as a function with respect to the filtration. Let the dimension d = 2, 3, and fix a direction v on the surface of the unit circle or sphere S^{d-1} (where v $\in S^{d-1}$). Let M_{d-1} be the set of all closed, compact subsets (shapes) embedded in $\mathbb{R}^d$ that can be represented in a finite, discrete manner as simplicial complexes [23]. Next, denote the simplicial complex representation of M $\in M_{d-1}$ by K, and let K_v indicate the v-orientation of K. The sublevel set filtration of K_v parameterized by a height function r(•, •) is the set $(x \in K : x.v \leq r)$. The v-directional parameter height function r_v (•, •) is
$r:K.S^{d-1} \Rightarrow R$
$(x,v) \rightarrow x.v$
Denote the external heights from this filtration by
$a_v:=\min(r_v(x),x\in K)$
$b_v:=\max(r_v(x),x\in K)$
We use the subscript notation to denote the simplicial complex representation K of a shape M, in the direction v, as K_v for d = (2,3). Similarly, we use the superscript notation K_v^x to denote the varying simplicial complex of K_v, generated by a sublevel set filtration with respect to Equation $k \in \mathbb{R}$ and defined by varying x $\in K_v$.
In considering a directional sweep over the surface of the sphere S^{d-1}, and calculating the corresponding EC curves χ_v^x of the finite simplicial complex representations K for every direction v $\in S^{d-1}$, the Euler characteristic transform (ECT) is defined as follows:
$ECT(K) : S^{d-1} \rightarrow \mathbb{Z}^R$

v$\to\chi(\mathrm{K}_v)$

In other words, the ECT of a shape collects EC curves of the shape, over all directions on the surface of the sphere.
We do this by smoothing a centered variant of function. The centered variant is given by taking the mean of curve over$\widehat{\chi}_v^x$ (a_v,b_v) and subtracting it from the EC χ_v^x (x) at every x $\in$ (a_v,b_v). This produces a centered EC curve in the direction v $\in S^{d-1}$,
Z_v^K:(a_v,b_v)$\to$R
$x\to$ χ_v^K-χ_v-K
We set the value of Z_v^K to be zero outside the interval (a_v,b_v) by default. Integrating the curve gives the following smoothed construct.
The centered, cumulative Euler characteristic curve or smooth Euler characteristic curve (SEC), for a fixed direction v $\in S^{d-1}$, is defined for all y $\in \mathbb{R} as$
$SEC(K)$:$S^{d-1}\to\mathbb{L}^2$
F_v^K(y):=$\int Z_v^K$(dx), integral between $-\infty$ and y
The smooth Euler characteristic transform (SECT) for a simplicial complex K of a shape M $\subset \mathbb{R}^d$,with d = (2,3), is the map
$SECT(K)$:S^{d-1} $\to\mathbb{L}^2$[a_v,b_v]
v$\to$ $\mathrm{F}_v^K(b_v)$

6. SECT vs PHT

One of the key advantages of the SECT over the PHT is the robustness of the SECT to choice of metric. As the brain is made up and connected of cerebral fiber pathways, the validity of the use of the usual Euclidean metric is brought in to question. Both volumetric and morphometric analysis requires the specification of a metric which in this case is extremely difficult. As the SECT is a topological property as opposed to being a metric-based property, like the PHT, it bypasses these difficulties. Using a topological property as opposed to a metric-based property allows comparisons of different stages of GBM without needing to account for time of progression.

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Predicting Clinical Outcomes in Glioblastoma: An Application of Topological and Functional Data Analysis
Glioblastoma Multiforme – Symptoms, Diagnosis and Treatment Options
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PROOF OF THEOREM: A METRISABLE SPACE IS COMPACT IF AND ONLY IF IT IS SEQUENTIALLY COMPACT.

CONOR MCCARTHY

Abstract. The reason I wanted to do a proof of the theorem in the title is because it has terminology that is relevant to the topology module MA342 but also expands a bit on certain definitions. The word 'compact' in a topological sense is covered in the module but the terms 'metrisable space' and 'sequentially compact' are not fully looked at, but are not so complex that students would be lost trying to understand them. This theorem can be found in Aisling McCluskey's book *Undergraduate Topology: A Working Textbook* in Chapter 4: Invariants.

Before beginning the proof of the theorem, I think it is important to briefly go over each term used in the theorem and define them, starting with a 'metrisable space' . A metrisable space is defined as a topological space that is homeomorphic to a metric space [3]. Essentially, this means that the topological space is comprised of the open sets from a metric space and the space obeys all of the separation axioms (Hausdorff, Regular, Normal etc.)[1]. The topological space will also have a metric.

Next up, we have the word 'compact'. Over the course of the MA342 module, the word compact has been defined in a topological sense but let me remind you again. A topological space is said to be compact when every one of its open covers has a finite subcover[1]. We have seen over the course of the module that, for example, $[0, 1]$ is compact with τ_{usual}, but $(0, 1)$ and $\mathbb{R}$ are not. Another example would be that any cofinite space (X, τ_{cf}) is compact as they obviously must have a finite subcover. A good way of imagining what compact means is that the space is not "fuzzy around the edges" [2].

Just briefly to explain, when a statement says 'if and only if' or 'iff' for short, the proof of the theorem must involve proofs going both ways. i.e. First prove a compact metrisable space is sequentially compact and then prove that a sequentially compact metrisable space is compact. ' $\implies$ ' is used to denote the first way and ' $\impliedby$ ' is used to denote the second way.

Now, lastly, we need to define what is means for a topological space to be 'sequentially compact'. A topological space (X, τ) is said to be sequentially compact if, in X, every sequence has a convergent subsequence (with the limit being in X) [3]. For metric spaces, the concept of compactness and sequential compactness are the exact same thing, but in topological spaces, there can exist compact spaces

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that are not sequentially compact, and vice versa[1]. An example of a space that is sequentially compact but not compact is the 'long line' with τ_{usual} and an example of a space that is compact but not sequentially compact is the product space $\{0,1\}^I = \{0,1\}^{[0,1]} = \prod_{[0,1]}\{0,1\}$ under the product topology [4].

For the proof of this theorem, I use two lemmas that the book *Undergraduate Topology: A Working Textbook* has stated and proven.

Lemma 1 Let S be any infinite subset of a compact topological space X. There is some $a \in X$ such that every neighbourhood of a contains infinitely many points of S [1].

Lemma 2 Given a sequentially compact metric space (M, d) and a positive real number δ, there is a finite family of open balls each of radius δ that covers M [1].

The only real new bit of terminology is the word 'neighbourhood' which is covered in MA342. Just to remind you though, for a topological space (X, τ), with an element p and subset N, N is said to be a neighbourhood of p if there is an open set G in τ for which $p \in G \subseteq N$ [1]. To give an idea as to what a neighbourhood looks like, a neighbourhood of an element x in a topological space (X, τ_{usual}) looks like an open ball with the x as its centre [1]. Now that you have the suitable background information, here is the proof of the theorem.

Theorem 0.1. *A metrisable space is compact if and only if it is sequentially compact.*

Proof. $\implies$ Let (M,d) be compact and let $(x_n)_{n\geq 1}$ be any sequence in M.

If any element of M occurs infinitely often in the sequence, its occurrences will form a subsequence which is constant, which is obviously convergent and hence, sequentially compact.

Let's say this is not the case, then $S = \{x_n : n \in \mathbb{N}\}$ is an infinite set. By **Lemma 1**, there exists some a such that every open ball $B(a, \frac{1}{k})$ contains infinitely many x_n's.

Then, choose n_1 so that $B(a, 1)$ includes x_{n_1}.
Choose, $n_2 > n_1$ so that $B(a, \frac{1}{2})$ includes x_{n_2}.
Choose, $n_3 > n_2$ so that $B(a, \frac{1}{3})$ includes x_{n_3}. And so on in this pattern.

Clearly, this is an infinite process and generates a subsequence $(x_{n_k})_{k\geq 1}$ for which

$$d(x_{n_k}, a) < \frac{1}{k} \to 0$$

as $k \to \infty$, meaning $x_{n_k} \to a$.

$\therefore (M, d)$ is sequentially compact.

$\impliedby$ Let (M, d) be sequentially compact and let $\{G_\alpha : \alpha \in I\}$ be a given open cover of M with no finite subcover. (i.e. Say M is not compact).

For each $n \in \mathbb{N}$, we can use **Lemma 2** to cover M by finitely many open balls of radius $\frac{1}{n}$. At least one of these open balls, say $B(x_n, \frac{1}{n})$ can't be covered by any finite number of G_α's as otherwise M would also be covered.
The sequence $(x_n)_{n\geq 1}$ thus created has a convergent subsequence $x_n \to l$.

This $l \in$ some G_{α_0} and there exists some $\varepsilon > 0$ such that,

$$B(l, \varepsilon) \subseteq G_{\alpha_0}$$

.

But, as $k \to \infty$ both $d(x_n, l)$ and $\frac{1}{n_k} \to 0$.

This means that for sufficiently large k, $B(x_{n_k}, \frac{1}{n_k})$ lie entirely inside $B(l, \varepsilon)$.

Therefore, they lie inside G_{α_0} which contradicts how the $B(x_n, \frac{1}{n})$'s were chosen. So, $\{G_\alpha : \alpha \in I\}$ has to have a finite subcover.

$\therefore (M, d)$ is compact $\square$

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THE POINCARÉ CONJECTURE

A PROBLEM SOLVED AND THE LIFE OF GRIGORI PERELMAN

DARREN O'BRIEN, EMILY GOTHARD AND MARIA KELLY

Abstract. The Poincaré conjecture is a topological problem established in 1904. It was not solved directly until the year 2002 by Grigori Perelman. The conjecture provided a unique way of characterising three dimensional spheres. Perelman had a distinguished career in mathematics and his work is well recognised worldwide.

1. Introduction:

In the year 2000, the Clay Mathematics Institute of Cambridge, Massachusetts in the United States proposed seven mathematical problems that were unsolved and offered a one million dollars prize to anyone who could provide a rigorous, peer reviewed solution to any of the problems. To this day only one of those problems has been solved, the Poincaré Conjecture, in 2002, by the Russian mathematician Grigori Perelman. Perelman decided not to accept the one million dollar prize money and not to accept the coveted Fields medal in 2006 [10]. Every four years the Fields medal is awarded for "outstanding mathematical achievement for existing work and for the promise of future achievement" [8] by the International Congress of Mathematicians. The Poincaré Conjecture is a mathematical topology problem about spheres. It was named after Henri Poincaré, a French Physicist and Mathematician who formulated it in the year 1904. A conjecture in mathematics is a hypothesis which is suspected to be true because of existing evidence and information, but for which no proof or disproof has yet been found [10].

2. The Precise Meaning of the Poincaré conjecture:

The Poincaré Conjecture states, in modern day language, that "if a compact three-dimensional manifold M3 has the property that every simple closed curve within the manifold can be deformed continuously to a point, does it follow that M3 is homeomorphic to the sphere S3?" [9].

3. The uncomplicated meaning:

In simpler terms this means, that any loop taken on the surface of a domain can be shrunk to a single point on a closed three dimensional manifold, a manifold is

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a topological space which is locally homeomorphic, meaning that there is a continuous one to one map from the open set to the ball Bn. A closed manifold is a manifold without boundary that is compact. This closed manifold is homeomorphic to the three dimensional sphere. Two spaces are homeomorphic if they share an Euler characteristic [2].

4. Initial framework of the conjecture:

Basically, the two dimensional sphere is a two dimensional surface seen inside a three dimensional space, that has the property that if an elastic band was wrapped around the two dimensional sphere, it would be possible to slide it down to a single point. Mathematically this means that any loop on the surface of the two dimensional sphere can be reduced to a point. Due to this property, the two dimensional sphere is said to be simply connected. However not every space has this property, donut shaped spaces are not simply connected because if a rubber band was wrapped around the donut once, it cannot be slid down to just one point without leaving the surface of the donut [6]. We know that this donut shaped space is called a torus. This property was known to be unique to the two dimensional sphere when the Poincaré Conjecture was formulated, because any other simply connected space that did not have any edges and was compact was actually the two dimensional sphere. Remember compactness is a property that generalises the notion of a subset of Euclidean space being closed and bounded [1]. It is now known that this is no longer true if we remove the idea of compactness since a circle together with its interior is simply connected, but an edge exists, the bounding circle.

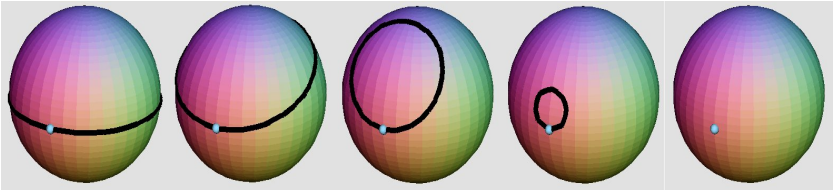

Figure 1. A compact two dimensional domain without boundary is homeomorphic to a two dimensional sphere if every loop can be continuously tightened to a point. The Poincaré conjecture proves that the same is true for 3-dimensional spaces [7]

.

5. Perelman's solution:

The Poincaré conjecture questioned if the same was corrected for the three dimensional sphere. The three dimensional sphere is a domain residing in four dimensions. Grigori Perelman proved the Poincaré Conjecture was indeed true by using concepts from topology and methods from geometry, meaning that the same was true for the three dimensional sphere [6].

6. The Poincaré Conjecture in higher dimensions:

The Generalised Poincaré Conjecture, is the Poincaré conjecture expanded to include higher dimensions. Amazingly it was much easier to prove this fact for dimensions greater than three dimensions. In the journal [15], it proved this fact to be true for the five dimensional sphere and the six dimensional sphere in 1960, long before it was proved for the three dimensional sphere. In the journal [3], it was proven that this fact was true for the four dimensional sphere. Freedman was awarded the Fields medal for his work on proving this fact for the four dimensional sphere.

7. New Ideas:

Perelman's proof of the Poincaré Conjecture established important techniques, which have been used over the years on other problems. The Poincaré Conjecture has led to greater understanding of three dimensional manifolds. Hamilton worked on proving the Poincaré Conjecture before Perelman, but wasn't successful. However both Hamilton and Perelman's work on the Poincaré Conjecture problem, developed a tremendous amount of attention and was of interest to many mathematicians and physicists studying other mathematical problems and theorems, one such example was Thurston's Geometrization Conjecture, which proved that certain three dimensional topological spaces have a distinct geometric structure associated with it [6].

8. The Man behind the Proof:

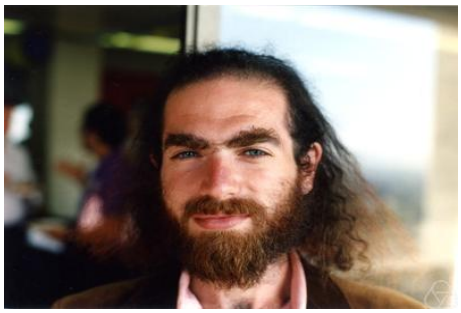

Figure 2. Grigori Perelman [16]

Grigori Perelman was born in 1966 in Russia to an electrical engineer and a mathematics teacher. After being taught English, Perelman was accepted into Leningrad's Special Mathematics and Physics School Number 239. As the name implies, this school was for talented students who were coined to have an exceptional ability and needed specialized education. At age 16, Grigori Perelman was selected as a member of the 1982 Soviet Mathematical Olympiad Team. To be chosen was an honor as the selection of this team was very competitive and involved a series of mathematical tests. Just by being on the team, Perelman was allowed automatic

acceptance into a university in Russia. When at the Mathematical Olympiad competition in Budapest, Perelman received a perfect score and was awarded a gold medal [11]. Perelman attended Leningrad State University to further his education. Graduating in 1987, he had already published four mathematical papers. Grigori Perelman applied for graduate school at Steklov Mathematics Institute in Leningrad. However, he was met with challenges as they did not accept Jews at the time, of which Perelman was. A recommendation from Aleksandrov, one of Perelman's advisors at Leningrad State University, allowed for an exception to be made and Perelman was admitted [11]. In 1992, he was awarded a postdoctoral fellowship at the Courant Institute at New York University. In 1993, he accepted a postdoctoral fellowship at Stony Brook at SUNY, the State University of New York before taking a two-year research fellowship at the University of California, Berkeley [4]. While in California, he proved many theorems and published many papers. Most significantly, however, was his paper proving the soul conjecture [16]. Additionally, he attended many lectures by Richard Hamilton. Hamilton had studied the Ricci flow [4]. After his two-year research fellowship had ended, Perelman returned to Russia and the Steklov Mathematics Institute [11]. Perelman never forgot Hamilton and eventually expanded the Ricci flow to use to prove the Poincaré conjecture in a paper published on November 11, 2002 [11]. After this paper was on the internet, he began to give lectures on it at Massachusetts Institute of Technology, SUNY at Stony Brook, Columbia University, and Princeton University [**?**]. He received many offers for professorship at universities and declined each one of them. Perelman published three installments of his paper on the Ricci flow: The Entropy Formula for the Ricci Flow and Its Geometric Applications [12], Ricci Flow with Surgery on Three-Manifolds [13], and Finite Extinction Time for the Solutions to the Ricci Flow on Certain Three-Manifolds [14]. These papers were reviewed and accepted as proof of the Poincaré conjecture and the Thurston Geometrization Conjecture [4].

9. Perelman Today:

In 2005, Perelman went public saying that he was resigning from mathematics. Perelman was awarded the Fields Medal, of which he graciously refused [4]. He has a history of declining recognition, like when he declined a European Mathematical Society prize in 1996 [11]. In 2010, Perelman was offered one million dollars, the prize given by the Clay Mathematics Institute for solving the Poincaré conjecture which he also rejected [4]. Perelman once said "If the proof is correct then no other recognition is needed" [5]. Perelman's ultimate reward was to solve the problem, fame and money were irrevelant. Nowadays, Grigori Perelman resides in St. Petersburg and is retired from mathematics [16].

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THE POINCARÉ CONJECTURE - EXPLANATION, ILLUSTRATION AND HISTORY

BEN O'CONNELL, LORCAN O'LEARY AND CONOR SHEEHAN

(Communicated by Prof. Graham Ellis)

Abstract.

In the early 20th century, Henri Poincaré introduced the concept of a theorem surrounding the characterization of the 3-sphere. He conjectured that every simply-connected, closed 3-manifold is homeomorphic to the sphere. In this paper, we explain the origin of the conjecture itself, explain it and illustrate its varied uses. This proof uses techniques such as the Ricci Flow and Reduced Volume. We explain these techniques and appraise their use in the proof. We then link the proof to modern mathematics and show its variety of uses with more well-known areas of math. [4] [6]

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History of the Proof

At the beginning of the 20th century, Henri Poincaré began laying foundations for topology as we know it today. From his appraised mathematical work in 1904 he described a counterexample to his own previous work in 1900 that homology was sufficient to tell if a 3-manifold was a 3-sphere. In the very same paper that the 'Poincaré Sphere' was introduced to topology, Poincaré begged this question which puzzled mathematicians for over 100 years. [1]

Consider a compact 3-dimensional manifold V without boundary. Is it possible that the fundamental group of V could be trivial,even though V is not homeomorphic to the 3-dimensional sphere?

This led to the Poincaré Conjecture as we know it today:

Every simply connected, closed 3-manifold is homeomorphic to the 3-sphere.

The conjecture itself gained a reputation for being extremely tricky to solve after numerous failed attempts in the 20th century by renowned mathematicians. Numerous false proofs were published and then retracted in this time frame also. To add to the frustration of solving the original conjecture, the notion of the conjecture was proved in every other dimension except for the third. [1]

It was not until Richard S. Hamilton introduced the 'Ricci Flow' in 1981 that any headway was made towards a way of tackling the proof. The Ricci Flow is an intrinsic geometric flow that shrinks directions of positive curvature and expands directions of negative curvature, while simultaneously smoothening out irregularities in the metric. Although Hamilton was successful in solving specific cases of the Poincaré Conjecture and despite extending his work he was unable to prove the conjecture. [1]

To add to the reputation that the Poincaré Conjecture already had, it was named one of the Seven Millenium Prize Problem, these were seven problems in mathematics in which The Clay Mathematics Institute offered a million dollar prize for per problem. [1]

It was not until 2002 and 2003 that a relatively unknown mathematician, Grigori Perelman, presented a proof of the conjecture in three papers made available in those two years. Adjustments were made to the proof in the following three years. Perelman was recognised to have solved the proof in 2006, 102 years after the conjecture was proposed. The proof of the conjecture was recognized as the scientific "Breakthrough of the Year", the first time it was given to the area of mathematics. [1]

Perelman, both a patriotic Russian and modest mathematician, refused all honours associated with solving the proof. These honours included the million dollar prize for proving the conjecture and the Fields Medal, regarded as one of the highest honours a mathematician can receive. He believed Hamilton played as large a role in the proof as he did, and therefore politely declined these accolades. He turned down multiple opportunities to give talks at famed universities all around the world, instead deciding to return to Russia and remove himself from the limelight. In 2005 Perelman quit his job at The Steklov Institute in Russia and, incredibly, is it is said that he has since abandoned mathematics entirely. [2]

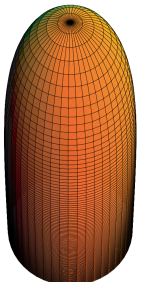
Explanation of Poincaré Conjecture

A complicated proof by nature, Perelman's proof of the Poincaré Conjecture utilizes a variety of mathematical areas. The following is a breakdown of the major steps involved in the proof and a dissection of key terms.

To begin, the manifold in question is deformed by the aforementioned Ricci Flow, which usually deforms the manifold towards a rounder shape. Manifolds are topological spaces which locally resemble real n-dimensional spaces. [3] However, the Ricci Flow encounters problems when presented with singularities, which occur during the deformation. Singularities are points at which a mathematical object is not defined or does not behave well e.g. a point that is not differentiable. These occur when the manifold is stretched apart from itself. Perelman realised he needed to understand these singularities before he could further his attempted proof. [1]

After significant time spent studying said singularities and creating a list of all possible singularities, Perelman concluded that they were all very simple and were simply three-dimensional cylinders made of spheres stretched along a line. Perelman proved this to be true using 'Reduced Volume'. The operation needed involved multiplication by an eigenvalue. An eigenvalue in this context was similar to a manifold playing a musical note. Perelman showed that the pitch of the note increases as the manifold is deformed via Ricci Flow. By proving this, he was able to eliminate the more tricky singularities that ailed him. By showing this, Perelman deduced that all strands and singularities can be cut and capped and none stick out on a given side. [1]

Perelman had been worried about a possible cigar solition forming a as a result of the Ricci Flow but managed to disprove its possible arisal and therefore could continue with the proof. The shape is seen below. [1]

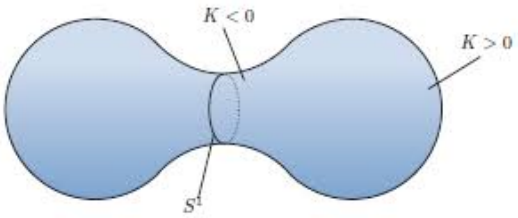


Perelman then took any compact, simply-connected three-dimensional manifold without a boundary and ran the Ricci Flow on it. Compact: implies closed and bounded. Simply connected space: this means that a given space has no holes i.e. a sphere [1] The manifold then deformed into round pieces with strands running between them. Strands can then be cut and manifold is deformed further until all that is left is a collection of three-dimensional spheres. The original manifold can then proceed to be rebuilt using these spheres and three-dimensional cylinders. The end result is a round shape forming, thus showing that the manifold is in fact homeomorphic to a sphere. [3]

Illustration

This proof has led to many new tools which can be used to help solve other problems in the mathematical world. The main problem Perelman faced was that in some cases the manifold was stretched apart from itself towards singularities, which are mentioned above.

An example of a manifold that stretched towards a singularity: where regions $k < 0$ tend to shrink and regions $k > 0$ tend to expand. This is how Perelman applied the Ricci Flow to the 3-manifold. [5]



He deformed the manifold until eventually he was left with a collection of round three-dimensional spheres. He then rebuilt the original manifold by connecting the spheres together with three-dimensional cylinders which morphs them into a round shape. This proved that the manifold was in fact homeomorphic to a sphere.

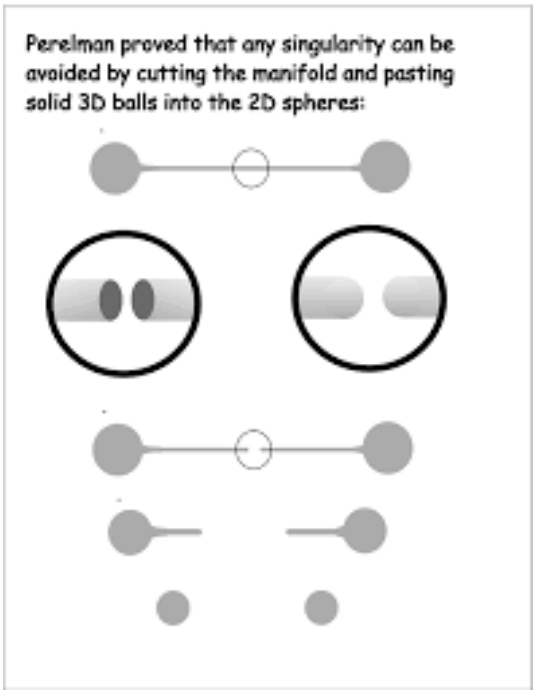


Image from [8]

Applications to Modern Mathematics

This breakthrough in the proof of this conjecture has helped mathematicians further understand other aspects of mathematics, such as Riemann surfaces.

The proof which was fully summarized on page 3 actually proved a stronger result, The Geometrization Conjecture:

That each of certain three-dimensional topological spaces has a unique geometric structure that can be associated with it.

This is an analogue of the uniformization theorem for two-dimensional surfaces. This theorem helps us understand that every simply-connected Riemann surface can be given one of three geometries (Euclidean, spherical, or hyperbolic). [7] [9]

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AMY FERNS, GENIVEVE MURPHY, AND SARAH MUTH

(Communicated by)

Abstract. This student paper will illustrate what is meant by the homology of a simplicial complex. This will entail giving the definition and examples of a simplicial complex, and what it is composed of. This paper will further investigate into the definition of homology and how it relates to the Euler Characteristic of a simplicial complex.

1. Simplicial Complex and Homology of Simplicial Complex

Definition 1.1. A compilation of simplices is known as a **simplicial complex**, denoted by K. This is true for:

(1) If K consists of of a simplex δ then all faces of δ is contained within K.

(2) If within K there are two simplices intersecting, then a face is created from the intersection of these two simplices.

Simplices, or also known as simplexes, are what essentially constructs a simplicial complex. Some examples of simplices are as follows:
(a) 0-simplex : a point
(b) 1-simplex : a line segment between two points
(c) 2-simplex : triangle
(d) 3-simplex : a tetrahedron
(e) 4-simplex : a 5-cell, also known as a pentachoron or a tetrahedral pyramid.

It is the connection of these simplices that create an simplicial complex. An example of how this is created is as follows:

Example 1.2. $\mathrm{K} = \big\{ \langle p0, p1, p2, p3\rangle, \langle p1, p2, p3\rangle, \langle p0, p2, p3\rangle, \langle p0, p1, p3\rangle, \langle p0, p1, p2\rangle, \langle p0, p1\rangle, \langle p0, p2\rangle, \langle p0, p3\rangle, \langle p1, p2\rangle, \langle p1, p3\rangle, \langle p2, p3\rangle, \langle p0\rangle, \langle p1\rangle, \langle p2\rangle, \langle p3\rangle \big\}$.

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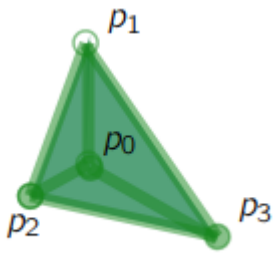

Figure 1. Graph 1

Now that we see how these simplices build a simplicial complex, we will look further into how these simplexes are created.

Definition 1.3. A simplcial complex, K, which contains all the sets of simplices contained in K, with a dimension equalled to or less than p is called a **p-skeleton**, expressed as K^p.

For example,
(a) K^0 represents the vertices.
(b) K^1 represents the vertices and the edges.
(c) K^2 represents the vertices, the edges and the triangles.
(d) K^3 represents the vertices, the edges, the triangles and the tetrahedrons.

Definition 1.4. If the simplex has a fixed order of vertices and is also a p-simplex, then it is known as an **oriented slimplex**, which is expressed as $\vec{\sigma}$. In this case, we would use square brackets [] instead of <> around the vertices that create the simplex.

Example 1.5. For example, we will look at the 2-oriented simplex, where the square brackets are used to represent the oriented simplex:

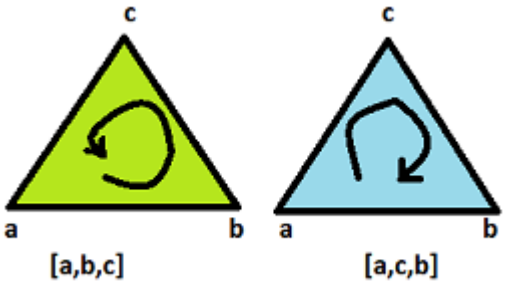

Figure 3. Graph 3

A p-simplicial complex can be formed when p-simplexes can be added with other integers, which leads to the formation of a **p-chain**.

Definition 1.6. In the case where all chains in the simplicial complex are oriented, this would be called an **oriented simplicial complex**. Also, where a simplicial complex had sub-simplices, this would also be known as an oriented simplicial complex, which is expressed as $\vec{\mathrm{K}}$.

Definition 1.7. When all sets of p-chains are created from an oriented simplicial complex $\vec{\mathrm{K}}$, this is known as a **simplicial complex chain**, which can be expressed as $Cp(\vec{\mathrm{K}})$.

Definition 1.8. Let $\partial = [p0, ...\hat{p}i, ...pn]$ be an oriented p-simplex. ∂ acts upon an oriented p-simplices as follows:

$$\partial[p0, ..., pn] = \sum_{i=1}^{n} (-1)^n [p0, ...\hat{p}i, ...pn]$$

$\partial[pi] = 0$
(1)

This is known as a **boundary operator** , where the oriented simplex $[p0, ...\hat{p}i, ...pn]$ is the i^{th} face of ∂, and is completed by taking $\hat{p}i$ out of the simplex.

Example 1.9. We can now see how the the boundary is formulated in the shape by the boundary operator. For example in figure 1.5:
$\partial[a, b, c] = (-1)^0[\hat{a}, b, c] + (-1)^1[a, \hat{b}, c] + (-1)^2[a, b, \hat{c}]$
= [b,c] - [a,c] + [a,b]
= [b,c] + [c,a] + [a,b]

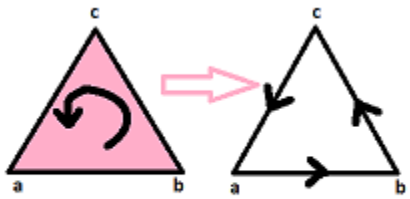

Figure 4. Graph 4

2. **Simplicial Homology**

Homology is a commutative alternative to homotopy. Let X be a topological space. $H_n(X)$ are the homology groups with n=0,1,2,... all commutative. $H_n(X)$ measures the number of n-dimensional holes in X.

Example 2.1. Let X be a graph with the vertices x, y, z and the edges a, b, c, d.

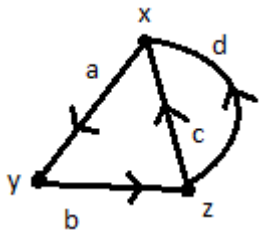

Figure 1. Graph X

The graph X has a loop abc that starts and ends at x, a loop bcd that starts and ends at y and a loop cab that starts and ends at z. Due to commutativity and writing the operations as addition, we get a+b+c = b+c+a = c+a+b. We call them cycles because the loop is geometrically closed.
Other examples for a cycle are: c-d and a+b+d.

Let C0 be a free abelian group of vertices x, y, z and C1 be a free abelian group of (directed edges) a, b, c, d. The elements of C0 are integral linear combinations of x, y, z (0-dimensional chains). The elements of C1 are integral linear combinations of a, b, c, d (1-dimensional chains). The edge's boundary is defined by:

$$\delta(edge) = finalpoint - initialpoint, \text{ e.g. } \delta(a) = y - x.$$

Definition 2.2. Let Cn be a group of n-dimensional chains. The chain in Cn is a cycle if its boundary is zero. And since $\delta^2 = 0$, we conclude that every boundary is a cycle.

Example 2.3. Let X be a graph (Figure 1) with vertices x, y, z and edges a, b, c, d. With each edge having a boundary:

$$\delta(a) = y - x, \delta(b) = z - y, \delta(c) = x - z, \delta(d) = x - z,$$

$\delta : C1 \to C0$ is the group homomorphism which extends:

$$\delta(\alpha a + \beta b + \gamma c + \zeta d) = \alpha\delta(a) + \beta\delta(b) + \gamma\delta(c) + \zeta\delta(d)$$

$$= \alpha(y - x) + \beta(z - y) + \gamma(x - z) + \zeta(x - z) = (-\alpha + \gamma + \zeta)x + (\alpha - \beta)y + (\beta - \gamma - \zeta)z;$$

$$\delta(a + b + c) = (y - x) + (z - y) + (x - z) = 0$$

So $\alpha a + \beta b + \gamma c + \zeta d$ is a cycle with -$\alpha + \gamma + \zeta = 0, \alpha - \beta = 0, \beta - \gamma - \zeta = 0$.

Definition 2.4. Let X be a topological space and let W be a sequence of k-groups, call them chains:

$$W : Ck \to ... \to C3 \to C2 \to C1 \to C0 \to 0$$

with

$\delta k : Ck \to Ck - 1, ..., \delta 3 : C3 \to C2, \delta 2 : C2 \to C1, \delta 1 : C1 \to C0, \delta 0 : C0 \to 0 = 0.$

The kernel of a boundary δn represents the group of cycles $\subseteq Cn$ is defined as:

$$Ker(\delta n) = Z(n)$$

The image of a boundary $\delta n + 1$ represents the group of boundaries $\subseteq Cn$ is defined as:

$$Im(\delta n + 1) = Bn)$$

Definition 2.5. Let the group Hn be a n-dimensional homology group of a simplical complex K over a topological space X. The relationship between the group of boundaries Bn and the group of cycles Zn defines the homology Hn .

$$Hn \equiv Zn/Bn \equiv Ker(\delta n)/Im(\delta n + 1)$$

Hn can be finite or infinite, e.g. 0, $\mathbb{Z}$, $\mathbb{Z} \oplus \mathbb{Z}$.

3. How does the homology of a simplicial complex relate to its Euler characteristic?

Definition 3.1. The Betti number is a topological invariant, it describes the number of n-dimensional holes in the surface
It is the rank of the n-th singular homology group

$$\beta n = rank(Hn)$$

Definition 3.2. The Euler characteristic describes the shape and the structure of a space regardless of the way it is bent.

Example 3.3. Using Let Cn be a group of n-dimensional chains again.
Cn denotes the number of n-simplexes in the complex. We know the euler characteristic is defined as the alternating sum of Cn

$$\chi(C) = \sum(-1)^n rank(Cn)$$

Example 3.4. We can also define the euler characteristic of a simplicial complex as the the alternating sum of its Betti numbers:

$$\chi(C) = \sum(-1)^n \beta n = \sum(-1)^n rank(Hn)$$

Conclusion

$$\chi(C) = \sum(-1)^n rank(Cn) = \sum(-1)^n rank(Hn)$$

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THE KNOT THEORY

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Abstract. This research paper is going to focus on Knots and Knot theory. We will discuss what a knot is by describing and displaying different types of knots. We will explain some of the benefits of knots and knot theory. Additional we will explore the origins of knot theory. From our research we distinguished the difference between a general "knot" in our shoe lace and the actual concept of a knot in knot theory.

1. Introduction

Knot theory first came about in 1771.It was then studied by multiple mathematicians during the 19th century where it has progressed to the concept of knot theory that we study today. From these studies we have identified unique strands and forms of knots. We have also looked at some of the many benefits of knots and knot theory in mathematics, biology, chemistry and physics.

2. What is a Knot?

The study of knots and their properties is known as knot theory. Firstly, the big question is what do we mean by a knot? We are all familiar with tying a knot in a rope or knotting our shoelaces. In this case we are considering similar knots but one thing we must keep in mind is that in our definition of a knot the two ends of the string much be attached in order to join a loop. From this once a knot is formed the string or loop cannot be untangled without breaking the string. (Adams, 2000) The formal definition of a knot is: "K is a knot if there exists a homeomorphism of the unit circle C into 3-dimensional space R^3 whose image is K. By the circle C is meant the set of points (x, y) in the plane R^2 which satisfy the equation $x^2 + y^2 = 1$." (Crowell Fox, 1997) As we tie our shoelace or look at a knotted necklace, we may think that every knot is the same but in fact there are various types of knots that are formed independently. It is not possible to rearrange or change one knot into a different type of knot. Some of the main types of knots that we are going to mention are the trivial knot, a trefoil knot, the figure-eight knot. (Adams, 2000)

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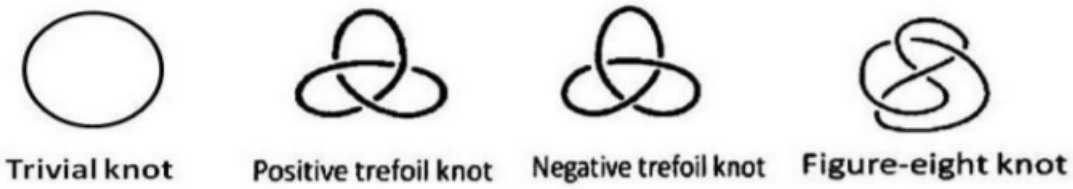
3. Identifying the difference between varies knot types



(Dehn, 1914)

As we have mentioned one type of knot in knot theory is the trivial knot which is also known as the unknot. This is said to be the simplest type of knot. It is a closed loop which is not knotted. It looks like a simple loop. It can not be changed into any other form of a knot without being cut or broken. "In the 1930s Reidemeister first proved that knots exist which are distinct from the unknot by inventing and making use of the so-called Reidemeister moves and coloring each part of a knot diagram with one of three colors." (Weisstein, 1999) Next we mentioned a trefoil knot. It is not possible to untangle a trefoil knot in order to make a unknot. Images of trefoil knots can look different from each other, although they can all be untangled to form each other. This shows us that they are the same type of knot. (Adams, 2000). When looking at trefoil knots there can be either a positive or negative trefoil knot. From figure 1 we can see that a positive trefoil knot and negative trefoil knot cross in opposite directions. These two types of trefoil knots are also known as the left-hand and right-hand trefoil knot. It was proved by Max Dhen that these knots are not equivalent using group theoretical methods. (Dehen 1914) The figure-eight knot is the final knot that we are going to look at. No matter what way the figure-eight knot is projected there will always be at least four crossings in the knot. A crossing is where the knot crosses over itself. (Adams, 2000) This is often the type of knot that climbers use.

4. What are the main goals of Knot Theory

The Knot Theory is a branch of Topology which studies knots to determine if they are equivalent. (Manturov, 2018) One of the main goals of the Knot Theory is to determine when two given knot diagrams verify the same knot. William Basener states that there are a number of ways to ascertain if knots are equivalent. One such way is by using the knot invariant, the minimal crossing number. This reveals that if two knot diagrams have individual minimal crossing numbers, they are distinct knots. Conversely if they have equal minimal crossing numbers they are equivalent. Alternatively, Tricoloring, where we can assign one of three colours, red, green and blue to each strand in a knot diagram such that at each crossing that all strands meeting are a different colour or the same colour (Basener, 2006). Adams however states that the most successful way of telling knots apart is by associating each knot with a polynomial. The polynomial is computed from the projection of the knot, but any two different projections of the same knot yield the same polynomial. Hence the polynomial is an invariant of the knot. (Adams, 2004) Research is ongoing in the various types of polynomial's to identify each type of knot correctly for example; the Alexander polynomial cannot distinguish all knots

from the trivial knot. Additionally, the HOMEFLY polynomial can be beneficial when determining the braid index of a knot (Adams, 2004).

The Knot Theory has been extremely beneficial to scientists in explaining the mechanism of how replication, transcription and recombination occur in Deoxyribonucleic acid (DNA). Brown and Cozzarelli (1979) applied the knot theory to investigating DNA. This provided cell biologists with quantitative, powerful and an invariant way to measure the properties of DNA. Principles of the knot theory helped to reveal the mechanisms by which enzymes unpack DNA. Furthermore, it has been instrumental in understanding the termination of DNA replication and the role of enzymes in recombination (M. Beals, 1999). Consequently, Adams stated that biologists investigating the Int enzyme, used by bacteriophage virus to produce a new DNA molecule by inserting its own DNA into a circular viral DNA molecule. Using their knowledge of the knot theory, biochemists synthesised single molecules and allowed the Int enzyme to act on them. However the knot theory couldn't fully explain this and it is hoped with more experimental data, mathematicians will be able to say precisely what topological effect the enzyme has (Adams, 2004).This will enable scientists to study DNA and gain a deeper understanding of how DNA is replicated and possibly provide answers as to why DNA replication can sometimes go wrong. Consequently advancing research in this field and possibly developing ways of preventing disorders in DNA replication.

The knot theory can be linked to statistical mechanics. Jones states that certain algebraic relations used to solve models in statistical mechanics were crucial to describing a mathematical property of knots known as a polynomial invariant. This link between the two passed through a theory intimately related to the mathematical structure of quantum physics. This theory is called von Neumann algebras, is differentiated by the idea of continuous dimensionality. Spaces typically have dimensions that are natural numbers; however in von Neumann algebras dimensions such as 2 or π are possible. This possibility for continuous dimensionality played a role in linking the two. In another direction, the knot invariants were found to occur in quantum field theory. Additionally Edward Witten has revealed that topological quantum field theory affords a natural way of expressing new ideas about knots (V.F.R, 1990). This will further develop their studies on knots.

It is evident from the above that the main goal of the Knot Theory is to advance study in the mathematical world as well as in biology, chemistry and physics by developing a better appreciation of the intricacy of the world around us and a deeper comprehension of how things work.

5. The Origins of Knot Theory

The idea of knot theory what first brought to light in the year 1771 by French mathematician Alexandre-Théophile Vandermonde. Vandermonde noted how certain topological characteristics could be found in the properties of knows in discussion of its geometrical positioning. He wrote a paper called "Remarques sur les Problemes de Situation". He Opened it with the following statement.

"Whatever the twists and turns of a system of threads in space, one can always obtain and expression for the calculation of its dimensions, but this expression will be of little use in practice. The craftsman who fashions a braid or net or some knots will be concerned, not with questions of measurement, but with those of position:

what he sees there is the manner in which the threads are interlaced" (Przytycki, 1992)

There were questions among mathematicians at the time of Vandermonde, questions about the place of knots in mathematics. It wasn't until later on into the beginning of the 19th century the mathematical study of knots began. Carl Friedrich Gauss created a linking integral for calculating the linking number for two separate knots. One of the oldest notes found among Gauss' belongings was a collection of knot drawings dated 1794. (Colberg, 2017) Gauss actually drew the universe of a knot. He came to the conclusion that if he labelled the points where the knots crossed A, B and C that a knot with n crossings would be classified as a sequence of 2n letters, called "the scheme of the knot" (Colberg, 2017). Although it was not until some of his later work involving electrodynamics was Gauss able to apply the mathematical concepts of knots. In this Gauss created the linking interval where he was able to calcutate the linking number of two separate knots.

Gauss' work inspired other mathematicians working in the field of topology to look at knots and knot theory. Mathematicians such as Johann Benedict Listing published papers of topology with large sections focusing on knots, Listing looked at knots and its mirror image. Listing later stated that the figure of eight knot and its mirror image are equivalent, or amphicheiral (Przytycki, 1992). Lord Kelvin also used knot theory in his theoretical work on the "vortex atom".

The big breakthrough however came later in that century when the work of three Scottish mathematicians led to the creation of the first knot tables. Peter Guthrie Tait, Thomas Kirkman and Charles Newton little. In 1867 Tait first started his work on the tabulation of knots and although Tait is recognised for this early tabulation of knots, it was mathematician Thomas Kirkman who made the first major contribution to the classification of knots (Colberg, 2017). Kirkman's sole focus of study for a long period was this tabulation for alternating knots. His tables included alternating knots with as much as eleven crossings. Then Tait partnered up Little to continue this work, they experimented with certain notations provided by the likes of Gauss and Listing as mentioned above. They finally came to a decision on an improvement of listings notation where it came across less ambiguous. The use of these tables led to Little and Tait going on to uncover more significant discoveries in the world of knots.

6. Conclusion

From our research we have concluded that the concept of knot theory has grew and advanced over the last two hundred and fifty years. It has developed from Vandermonde's original idea of twists and turns of a system of threads. Today knots have become very useful and are identified in many ways and types as we have illustrated above. We are looking forward to seeing the progression of knot theory in the future.

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Women in Topology

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Abstract

Topology is the study of spaces or shapes that remain unchanged under continuous deformations such as bending and stretching. In this field of mathematics, many of the most known and celebrated mathematicians are male. For that reason, this article aims to highlight the work of a selected three female topologists, by outlining their background and their contributions to topology.

Keywords: Topology, women, Beloch, Salicrup, Kuperberg, WIT-HT

1 Margherita Piazzola Beloch (1879 - 1976)

1.1 Background

Margherita Piazzola Beloch was born on 12 July 1879 in Frascati, Rome. Her father, Karl Julius Beloch, was a classical and economic historian who taught at Sapienza University of Rome for 50 years. Margherita Beloch studied mathematics at Sapienza University. She wrote her undergraduate thesis, *On Birational Transformations in Space*, under the supervision of Guido Castelnuovo. Castelnuova is best known for his contributions to algebraic geometry. Beloch's undergraduate thesis was later published in *Annali di Matematica Pura ed Applicata*, one of the oldest Italian mathematical journals. Later she became a professor at the University of Ferrara, Italy. [1]


[1]

1.2 Contributions

Beloch worked mainly in algebraic geometry, algebraic topology and photogrammetry, studying the configurations of lines and rational curves that could lie on surfaces. She obtained the result that *"Hyperelliptic surfaces of rank 2 are characterised by having 16 rational curves."* She continued working on the topological properties of algebraic curves for most of her life.

Beloch is known for using her background in topology, along with geometry, to contribute to the world of paper folding, or origami. She became the first person to show and prove that the real roots of any cubic polynomial can be found using origami. She formalised the origami move which finds a simultaneous tangent to two parabolas. [2]

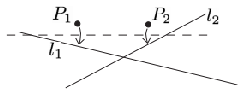

Figure 1: The Beloch origami fold. [2]

Other topological problems that interested Beloch included the mathematical analysis of photographs, particularly aerial photographs and X-rays. At the age of 86, a selection of her articles on photographic analysis and topography were published. Beloch died in Rome, Italy in 1976, at the age of 97. [1]

2 Graciela Salicrup (1935 - 1982)

2.1 Background

Graciela Beatriz Salicrup López was born in Mexico on April 7, 1935. She attended the Colegio Alemán, which is a German language primary school based in Mexico city. Graciela then went on to attend secondary school at a religious college for women. She graduated with a degree in architecture from the National Autonomous University of Mexico (UNAM) in 1959, but still wanted to become a mathematician. She enrolled in the Faculty of Sciences in 1964 and taught mathematics between 1966 and 1968 at UNAM before graduating in 1969. [3]

2.2 Contributions

Graciela's main focus was concerned with the structure of the topology category of topological spaces with continuous functions. She was a key pioneer in the field of categorgical topology. She contributed many ideas and results which feature in the paper; Factorizations, denseness, separation, and relatively compact objects [4] These findings were prepared and compiled into an unfinished manuscript by Salicrup before her untimely death on July 29, 1982. She is known for her pioneering of categorical topology.

The paper consists of the relationships among topological notions, such as, 'Hausdorff', 'perfect', 'closed', and 'compact', and how they are abstracted to more general settings, where they remain intact. It investigates factorization structures (focuses primarily on single morphisms) and their particular relationship to strong limit operators. They also focused on the Pumplün-Rörl Galois correspondence between the classes of morphisms and the classes of objects in any category. Internal characterizations of Galois-closed classes are provided as well as many more examples. [4]

3 Krystyna Kuperberg (1944 -)

3.1 Background

Krystyna M. Kuperberg (neé Trybulec) was born to pharmacist parents. She was brought up in Tarnów, Poland. Kuperberg attended the University of Warsaw in 1962, where she studied mathe-

matics. Her older brother, Andrzej Trybulec, also attended the University of Warsaw, and studied mathematics and computer science. [5]

In university, she attended topology lectures taught by Karol Borsuk, where her attraction to topology began. This is also where she met her husband, Włodzimierz Kuperberg. After getting her degree, Kuperberg continued her topological career by doing a masters, working closely with Borsuk. However, she did not continue straight onto a PhD, as she, her husband and two children moved to Sweden in 1966. In 1972 they then moved to the United States, where they have resided since. In 1974 Kuperberg received her PhD in Rice University, Texas, working under William Jaco. Also in 1972, Kuperberg and her husband both became faculty members of Auburn University, Alabama. [5]

3.2 Contributions

She and her husband collaborated on many papers together, one being *On weakly zero-dimensional mappings* in 1971. They proved that Cantorian manifolds could be mapped by a weakly zero-dimensional mapping onto a space of lower dimension, originally proposed by A. Lelek. Kuperberg also published solo, which included *An isomorphism theorem of the Hurewicz-type in Borsuk's theory of shape* [6] in 1972. She expanded on this publication in 1975 with *A note on the Hurewicz isomorphism theorem in Borsuk's theory of shape*.

However, Krystyna's most celebrated contributions to topology was published in 1994, in *A smooth counterexample to the Seifert conjecture* in *An Annals of Mathematics* [7].The Seifert Conjecture was published in 1950. She and her son, Greg Kuperberg, who also became a mathematician, then expanded on this work in 1996 with *Generalized counterexamples to Seifert Conjecture*, improving on the first publication.

3.3 Recognition

Kuperberg received the Alfred Jurzykowski Prize for translating mathematical works from Polish into English in 1995. Also, in 1996 she was appointed on the Council of the American Mathematical Society. [5]

4 Women in Topology-Homotopy Theory (WIT-HT)

Women in Topology-Homotopy Theory (WIT-HT) is an organisation which aims at the retention of women in the field of topology. Women in Topology (WIT) was the first workshop organised by WIT-HT, to increase visibility of female topologists, students and researchers. They worked in groups of 5-7, and many of them are still collaborating together on projects from these workshops. [8]

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A CLOSER LOOK INTO HOMOTOPY

CATHAL CONNELLY, RONAN O'MALLEY, AND SEAMUS SMITH

Abstract. The concept of Homotopy is one we came across in our Topology course. This document looks to explore Homotopy and related notions as well as expanding the topic beyond 'homotopy equivalence'; the 'homoptopy' boundary of MA342 if you like.Homotopy,aside from our mathematical definition gives us a way to interpret whether or not a 'continuous deformation' exists between the maps going from one topological space to another.

1. An unseen example in Euclidean planes

f:R$\rightarrow R^2 : f(x) \rightarrow x^3$
g:R$^2 \rightarrow R : g(x) \rightarrow e^x$
H: X x [0,1]$\rightarrow Y = Rx[0,1] \rightarrow R^2$
Defined by H(x,t): (x,t)$\rightarrow 1 + (1-t)x^3 + te^x$
$\rightarrow H(x,0) = x^3 = f(x)$
H(x,1) = e^x
f $\simeq g; \rightarrow$ This implies that H(x,0) and H(x,1) = g(x)
And hence, the two maps are homotopic.
This example illustrates the difference in the notions of a homeomorphism between two spaces and a homotopy.
As seen in class, R is not homeomorphic to R^2.
due to the number of connected components being a topological invariant under the restrictions. A homotopy does exist however.

2. An Ironic Example

Let the body of the capital letter 'X' be the topological space "X". Let the body of the capital letter 'Y' be the topological space "Y".
Let f: X $\rightarrow$ Y. Map the points (1,2,3,4,5) onto the 'Y' space, with 3 of 'X's' prongs going to the 3 prongs of 'Y', while the point at the centroid of 'X' along with the remaining prong are mapped to the intersection 'Y's' 3 prongs.
Let g:Y $\rightarrow$ X map all of 'Y's' points onto the 4 prongs of 'X'.
f and g are both clearly continuous, satisfying the minimum requirement for both homeomorphisms and homotopies. However f is a surjection and not an injection and is therefore not bijective. g is an injective but not surjective. Therefore neither maps are bijection and it follows that no such homeomorphism exists.This can be shown by the number of connected components each space has under the restriction of removing the point at the centroid of 'X' and it's corresponding

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map in Y. This will leave X with 4 connected components and Y with at most 3, depending on the map you choose and so the two spaces are not homeomorphic to each other as the number of connected components in each space is a topological invariant.
However the composite function f o g is homotopic to the identity function on 'Y' and g o f is homotopic to the identity function on 'X'.Although homeomorphism implies homotopy equivilance, the converse is not always true.

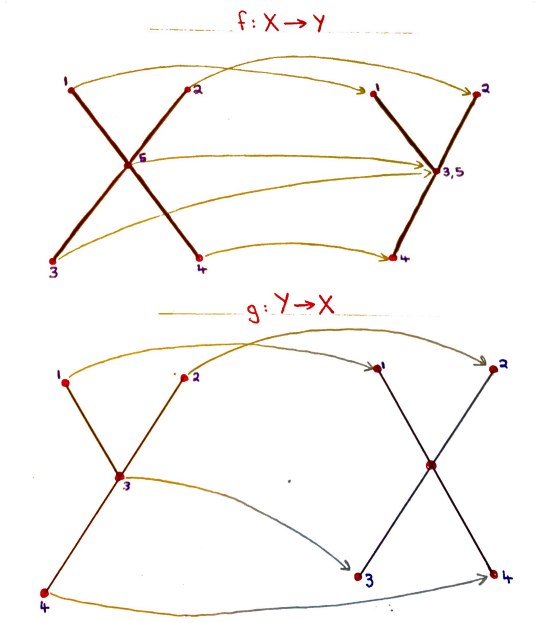

FIGURE 1

The two spaces are therefore homotopy equivalent. The brings along the concept of a "Contractible Deformation". Due to this space being contractible, it can be continuously shrunk into a point within the respective space.

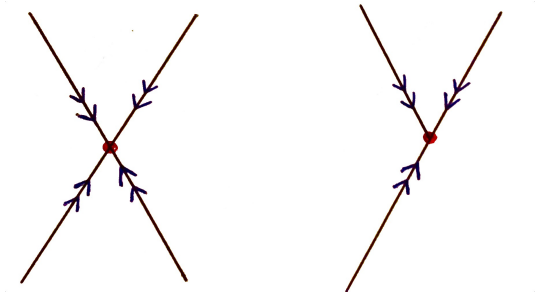

FIGURE 2

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3. Contraction of a Cone

Here we let the radial cone be the topological space X with the point 'x' at its apex. The topological space Y is the point 'x'. A 'Continuous Map' is 'Null-Homotopic' if it is homotopic to a constant map.

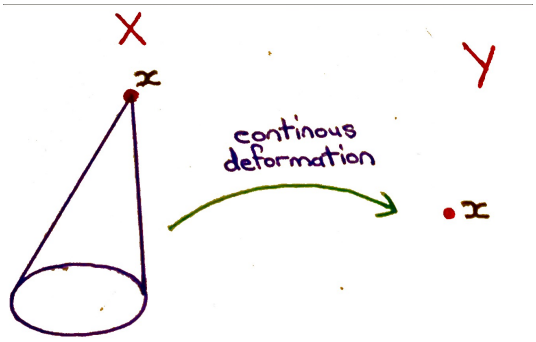

Figure 3

Intuitively one would think that this map is a "Deformation Retract", by viewing the above diagram. ie. The cone can be shrunk to its apex, by continuously deforming its structure. This is correct and is given by the following continuous map, H:X*[0,1]$\rightarrow Y : x \rightarrow (x, (1-t)s)$.
When X is compact and Hausdorff, when X is embedded on the Euclidean Space, that X can be visualized as the collection of lines joining each point of X to a single point, in this case x.
On a side note, we observe that both spaces have the same Euler Characteristic.
$\chi(Cone) = \chi(x) = 1$
This is what we would expect. The Euler characteristic of a topological space is a homotopy invariant, hence, will not change under any continuous deformation of the cone as both are homotopy equivalent.

4. Null Homotopy in the Complex Plane

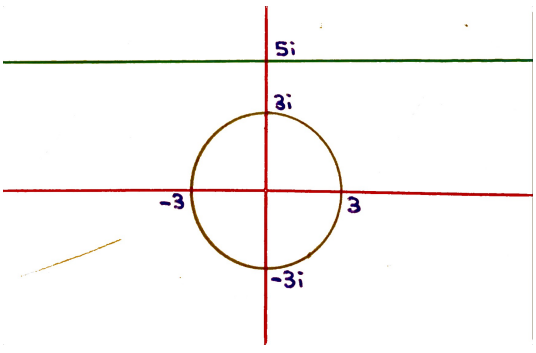

Figure 4

Now we visit the complex plane, which may bring back memories of Complex analysis (MA2287).
f: C$\rightarrow C, X \rightarrow Y : 3e^{i\theta}(0 \leq \theta \leq 2\pi)$.
This is the circle with centre at (0,0) and radius of 3 in the complex plane.
Define the constant function g: g$(\theta) = 5i$
Define the continuous map H$(\theta, t) : X*[0,1] \rightarrow Y : (\theta, t) \rightarrow ((1-t)*f(\theta))+(t*g(x))$
H$(\theta, 1) = 3e^{i\theta} = f(\theta)$
H$(\theta, 1) = 5i = g(\theta)$
And so f $\simeq g$,
Therefore, since g is a constant function, a Null Homotopy exists between f and g. This is as one might expect as visually it seems plausible that one could continuously deform the boundary of a given circle into a line in the copmplex plain.

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EXPLORATIONS IN KNOT THEORY

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(Communicated by Milan Lukic)

Abstract. Knot theory is a branch of topology which studies continuous closed loops, known as knots. Inspiration for the field of knot theory has come from artistic design; with knots being a feature of the traditional art styles of several historical cultures, perhaps the greatest example of which being the Celts.

In addition to art, Lord Kelvin once postulated that atoms were knots formed from aether. While this initial hypothesis has since been proven to be false, knot theory has developed as a field, and is now used commonly in advanced contexts in the physical sciences, such as string theory in physics, mapping organic compounds in chemistry and in DNA analysis in biology.

Knot theory is a relatively straightforward topic to understand, so long as a small number of prerequisite concepts are understood; such as ambient isotopy and knot invariance. In this paper, we set out to offer a beginner's introduction to knot theory, discussing with the reader a number of notations and key concepts within the area.

Introduction

Knot theory is the study of 3-dimensional closed curves, and the deformations possible without cutting through each other (in other words, topological deformations). The two questions are whether it is homologous to an unknot, or a circle, and the second is whether two knots represent different knots or are they deformable into each other. If they DO represent each other, then they are both representative of a prime knots[1]. Projecting it onto a plane and counting over and under crossings is a way. Measuring its complexity by seeing the least number of crossings that occur; the more crossings there are, the more distinguishable knots there are also.

Besides topology, knot theory has shown uses in quantum field theory, hyperbolic geometry, cosmology and biology chemistry and mathematical physics

Alexander Briggs Notation

The typically traditional notation, devised in the 1927 James W.Alexander and Gardland B.Briggs paper. Later iterated upon by Dale Rolfson when he created his knot table.

This notation is pretty simple. It organizes knots by their crossing number. Write the crossing number, superscript it to denote a component number and subscript it to denote its order within the links with the same number of components and crossings.

(Note that order is arbitrary, although in each number the twist comes before

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the torus)

Links are written with the crossing number superscripted to denote # components and subscripted to denote its order.

EG. Trefoil

$$3_1$$

and the Hopf link is

$$2_1^2$$

Dowker Notation (aka Dowker-Thistlethwaith notation)

This code is shown as a finite sequence of even integers. We generate these numbers by following the knot and marking crossings with integers.

As every crossing gets two visits, we create odd-even pairs of integers.

We then give each number an appropriate sign, + for over and - for under. In doing this we create a list of crossings, labelled with pairs. We then take the second number in each pair and turn these into a sequence, which is the "Dowker Notation".

However, it is accepted that a knot diagram can have multiple Dowker notations, and it is because of this that there is an understood ambiguity when reconstructing knots using their Dowker Notations.

Conway Notation

Conway notation is for both knots AND links, named after John Horton Conway (based on his theory of tangles (1970)). This notation is advantageous because it reflects some properties of the knot.

The notation describes how to construct link diagrams. We start by taking a basic polyhedron, which is a 4-valent connected planar graph with no di-gon regions. We take this polyhedron and we denote it first by the number of vertices, and then we denote the number of asterisks which determines the polyhedrons position on the list of polyhedra. For example, 7* denotes the first 7-vertex polyhedron on Conway's list. We then substitute into it an algebraic tangle at each vertex (note that all vertexes are orientated so there is not an arbitrary choice). All of these tangles have notation made up of + or - signs. We separate each tangle with a period in our data.

We can build very compact notations even for large crossing numbers. We also have further shorthands, such as for an algebraic knot we omit the 1*.

Conway's paper broke new ground in knot theory lists 10-vertex basic polyhedra that he uses to tabulate links, which is also the new standard for links of this nature. However, when we break into higher vertices, we are forced to use nonstandard choices.

Gauss Code

Gauss code is very similar to Dowker notation, representing a knot with a sequence of integers. The difference between Dowker and Gauss is that rather than representing each crossing with two numbers, as in Dowker, we display each crossing with 1 number, and a sign. If the sign is -, it is an undercrossing, otherwise it is an overcrossing.

Gauss code (as with any type of code) it is limited by its ability to identify knots. The point at which we begin to trace the knots crossings is arbitrary. The direction that we begin to trace crossings in is also arbitrary.

Gauss code is also unable to indicate the handedness of each crossing. For example, we cannot use the Gauss code of a trefoil knot to distinguish between left-handed and right handed crossings.

Advanced Gauss Code

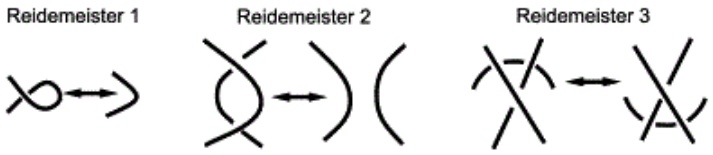
However, the nature of the indistinguishable nature of Gauss code can be resolved by using extended Gauss code. This modification of Gauss code is that the second instance of every number, rather than using +/- to represent over/under crossing, it is used to indicate the handedness of that crossing. Right-handed is + while left-handed is -.

Reidemeister Moves

Reidemeister moves, also known as equivalence moves, were developed in Knot theory in the 1920's, independently developed by both James Waddell Alexander and Garland Baird Briggs in 1926, and later Kurt Reidemeister in 1927.

Reidemeister moves began as a way of establishing equivalence in knots. That is, when a diagram of one knot can be continuously deformed to arrive at a second knot diagram, thus showing that the two seemingly different knots are actually homotopic.

The key idea of a Reidemeister move is that any knot deformation can be boiled down to one of 3 different sequences and represented on a link-diagram as one of the following:
TWIST - A twist (or untwist) in one direction (or the other).
POKE - Moving one loop completely over another loop.
SLIDE - Move a string completely over or under a crossing



To arrive at this conclusion, there are a few points that must be understood. The first idea we must be familiar with is that of an ambient space, which is essentially the space surrounding an 'object', which from now on we will refer to as an 'embedding'. For example, if we were to consider a 1-dimensional line 'L', this can be studied by itself, hence giving us the ambient space being 'L'. Alternatively, we can consider the space it is embedded in, which is R-2, which would mean an ambient space of R2.

To state the idea of knot equivalence formally[2]; two knots K1 and K2 are equivalent if there exists a continuous mapping

$$H : R^3 \times [0,1] \implies R^2;$$

such that for each $t \in [0,1]$, the mapping taking $x \in R^3$ to $H(x,t) \in R^3$
is a homeomorphism of R3 onto itself.

$$H(x,0) = x \; \forall \; x \in R^3 \wedge H(K1,1) = K2$$

From the idea of ambient space, we can draw the idea of an ambient isotopy, which is similar to isotopy where the ambient space is distorted along with the embedding that is being studied[3]. An ambient isotopy is a continuous deformation of such an ambient space. Upon these notions, Reidemeister developed his theorem:

"If two links are piecewise linearly equivalent (ambient isotopic), then there is a sequence of Reidemeister diagram moves taking a projection of one link to a projection of the other."

This was a major development in knot theory, as it provides a framework for unknotting complicated knots in order to assess whether they are isotopic to a pre-existing knot. This is particularly helpful when seeking to establish whether or not such complicated knots are equivalent to existing prime knots which have been tabulated over the years.

Reidemeister moves offer a framework for establishing if two knots are the same – if you can deform from one knot into another knot with a finite number of Reidemeister moves, then the knots are ambiently isotopic – i.e. they are the same knot. However, this is only one half of the battle. We have a method for confirming if two knots are the same, but how can we confirm that they are not the same?

To prove that two knots are not the same using Reidemeister moves is one idea, however this would rely on a proof by exhaustion – we might never arrive on a finite number of Reidemeister moves which would confirm that two knots are definitively unrelated. We could continue down a rabbit hole, applying Reidemeister moves ad nauseum, and getting no closer to the result we sought. This paves the way for our next concept – knot invariance.

A knot invariant is a quantity that is the same for two knots that satisfy knot equivalence. That is, although their diagrams appear differently, if they are in fact the same knot, their knot invariants will be equal.

There are a variety of ways to compute such a non-equivalence, such as the Jones polynomial or the Alexander-Conway polynomial. However, for an introductory paper, the most intuitive knot invariant is something called tricolourability. For a knot to satisfy tricolourability, there are a few rules it must meet:
1. At least two colours must be used, and
2. At each crossing, the three incident strands are either all the same colour or all different colours.

Therefore, when one knot is tricolourable and another is not, we know for certain that they aren't equivalent knots.

Trefoil Knots

The trefoil knot is the simplest version of a nontrivial knot and can be formed by joining the loose ends of an overhand knot. It has three crossings and is named after the clover (trefoil) plant. Since it is nontrivial, the trefoil knot cannot be untied, and can only be broken by being cut. The trefoil knot is also a (3,2)-torus knot or a (2,3)-torus knot.

It is a prime knot and using the Alexander-Briggs notation – used to distinguish between the different knots containing the same number of crossings – is denoted 3_1 and it cannot be deformed into its mirror image[4]. The trefoil knot is also the basis of the two most simple composite knots: the granny knot and the square

FIGURE 1. Exaple of a Left-Handed Trefoil knot

knot, which are the mathematical versions of the common granny knot and reef knot respectively. Both of these knots are formed by connecting two trefoil knots together. The trefoil knot is 3-colourable which means that if you colour the knot with three colours, in this case red, blue and green, at each crossing a colour either meets itself or all three of the colours meet.

The trefoil knot can also be made by putting 3 half-twists in a Möbius band and cutting down the middle. With a Möbius trefoil knot, in order to trace the knot from one side and end up where you started, you have to trace the knot twice whereas with a normal trefoil knot by tracing it from one side once you will end up where you started.

Another interesting feature of a trefoil knot is if you have a thick band forming the knot and cut out the middle of it, you are left with a Möbius style knot where each of the three crossings is split into four crossings and you end up with twelve crossings. This type of knot is sometimes referred to as a 'knot divided', and in 2005 it was made into a snow sculpture by Carlo Séquin – from the University of California, Berkeley – and four others.

The trefoil knot was known as a triquetra in ancient times, derived from the Latin words "tri" and "quetrus" which mean "three" and "cornered" respectively.

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INTRODUCTION TO THE ORIGINS AND GOALS OF KNOT THEORY

SHANE DOYLE, SHANE LAWLESS, AND COLLETTE O'SHEA

(Communicated by David Futer)

Abstract. In this Paper, we explore Knot Theory, a topic which is outside the curriculum of the MA342 module in Topology. We investigate where it originates from by tracing through key developments in it's fascinating history. We then explain and elaborate on what Knot Theory is, and define some basic concepts within it. We also look at what it's main goals are, along with various examples of its modern applications.

1. The Origins of Knot Theory

1.1. **Ancient Roots.** Humans fascination with knots dates back ancient times. They had many practical uses such as sailors using them to moor their boats to land. They are also used as decorations across disparate cultures. Intricate knots can be found from Celtic artwork such as The Book of Kells to that of the "endless knots" of Tibetan Buddhists. One of the earliest surviving documents on the applications of knots was by the Greek physician Heraklas, who wrote step-by-step instructions of how to tie an orthopaedic sling in 18 different ways.

1.2. **Geometria Situs.** Gottfried Leibniz's work may be considered the precursor to Knot Theory. In 1679, he pontificated about the need in maths for the study of "geometry of position" which deals with relations depending on position alone, without worrying about magnitudes. An example of this would be the famous seven bridges of Königsberg, with Euler's paper (1736) on the subject commonly considered the root of both the fields of graph theory and topology.

1.3. **Early Mathematics of Knot Theory.** It took nearly a century until in 1771, the formal study of knot theory in mathematics began to take place. Alexandre-Theophile Vandermonde wrote a paper "Remarques sur les problèmes de situation" ("Remarks on problems of positions") in which he uses braids and knots in the field of geometry of position.

Carl Friedrich Gauss also showed interest in Knot Theory. In his notes starting from 1794, he drew a variety of knots and illustrated a method of coding the crossings in a knotting. In 1833, he also introduced the notion of the linking number of two knots, which represents the number of times that each curve wraps around the other and shows if two knots are linked or not (linking number $= 0$ -not linked, linking number $\neq 0$ -linked).

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Johann Benedict Listing (1808-1882), a student of Gauss, published the article "Vorstudien zur Topologie" with a significant portion dedicated to knots. Notably, he showed that left and right trefoil knots are not equivalent, but that the mirror images of the figure eight knot were.

Henri Poincaré was the first person to solve the problem of distinguishing non-equivalent knots in his "Analysis Situs" paper in 1898, which laid the foundations for Algebraic Topology.

1.4. **Physics and Knot Theory.** In 1867, Scottish physicist Peter Tait conducted an experiment which produced vortex smoke rings which inspired another physicist, Lord Kelvin, about the possibility of vortex atoms. Tait in turn used Lord Kelvin's paper to help understand the structure of knots by the number of their crossings. Tait would go on to work with Thomas Kirkman to create a table of prime knots of up to 10 crossings. He would also introduce three basic principles for creating these knot tables.

2. The Mathematics of Knot Theory

2.1. **Basic Concepts in Knot Theory.**

Definition 2.1. A knot K is simple continuous closed curve in $\mathbb{R}^3$ consisting of a function

$$f : [0, 1] \to \mathbb{R}^3$$

With $f(0) = f(1)$ to form a closed loop
and with $f(x) = f(y)$ implying $x = y$, or $x = 0 \ \wedge \ y = 1$, or $x = 1 \wedge y = 0$
More Generally:
A subset K in a space A is called a knot if it is homeomorphic to a sphere S^n

Definition 2.2. A knot projection is called a regular projection if no three points on the knot project to the same point, and no vertex projects to the same point as any other point on the knot. A knot diagram is the regular projection of a knot to the plane with broken lines indicating where one part of the knot under crosses the other part. Informally, an orientation of a knot can be thought of as a direction of travel around the knot

Theorem 2.3. *If 2 knots K & J have the same knot diagram they are equivalent*

Definition 2.4. A link is a finite disjoint union of knots $L = K_1 \cup \ldots \cup K_n$. Each knot K_i is a component of the link. $\mu(L)$ or the multiplicity of the link is then number of components in a link. A knot K is a link such that $\mu(K) = 1$

2.2. **Reidemeister Moves.** A knot can have any of 3 Reidmeister Moves applied to it which replaces a simple configuration of arcs & crossings in a disc by another configuration. A move of Type 1 inserts or deletes a 'kink' in the diagram, moves of Type 2 uncross 2 lines & moves of Type 3 preserves the number of crossings.

2.3. **Knot Equivalence.**

Theorem 2.5. *Any 2 diagrams of equivalent knots are related by a finite sequence of Reidemeister moves*

Similarly 2 knots K_1, K_2 are equivalent if there exists an isotopy

$$h : \mathbb{R}^3 \times [0, 1] \to \mathbb{R}^3$$

such that $h : (K_1,\ 0) = h_0(K_1) = K_1$ & $h : (K_1,\ 1) = \ h_1(K_1) = \ K_2$, these knots are referred to as ambient isotopic.
An ambient isotopy is required as a normal isotopy as all knots can be continuously deformed to a point through a process known as Bachelors' unknotting hence meaning all knots are isotopic to the trivial knot.

2.4. **Prime Knots.** A knot K is said to be the unknot if it bounds an embedded piecewise linear disk in the 3-sphere S^3, this is equivalent to the boundary of a single 2-simplex linearly embedded in S^3, hence it has a diagram with no crossings

2 oriented knots K_1, K_2 can be added to form the sum $K_1 + K_2$ by a method corresponding to the intuitive idea of tying 1 & then the other in the same piece of string. This addition is commutative & can also be seen to be associative, with the unknot considered as the identity element in this addition. An example of this would be the addition of the left & right trefoil knots to form the Granny Knot

3. Main Goals & Applications of Knot Theory

Knot theory, as abstract and extraneous as it appears at first glance, is one of the most relevant and remarkable field of mathematical research in the 21st century. It aims to answer of; what is a knot? How can one tell them apart? How many types are there? Simple questions with complex solutions and astounding implications. The consequences of these implications have had an immense effect on numerous fields. From physics to encryption, the study and understanding of knot theory has enabled breakthroughs and research that would have previously been thought impossible.

3.1. **DNA Unpacking.** Arguably, the most well-known example of knot theory in practice is in the field of biology. Specifically, DNA, the genetic material that contains all the information to build and maintain an organism. One of the main purposes of DNA is to replicate cells. However, DNA is tightly packed into genes and chromosomes, so in order to replicate itself, it first must unpack itself. DNA packing is visualised as two stands that have been intertwined millions of times, tied into knots and subjected to successive coiling. However, replication is much simpler to achieve if the DNA is neatly arranged rather than tangled up in knots. This is where enzymes are essential to unpacking process. Enzymes act to slice through individual knots and reconnect strands in a more orderly fashion, creating easy analysable knots.
Biologists have used knot theory to gain insights into the DNA unpacking process. The two mainstreams ways they accomplish this is to use electron microscopes to take photos of DNA. A protein coating lets underlying and overlying segments to be distinguished. From this the unknotting number and crossing number can be estimated. The more prevalent method for calculating the crossing number is using a gel electrophoresis. The crossing number and the distance DNA fragments move on an electrophoretic gel is highly correlated. From this the ideal knot number can be estimated from a graph. These methods help show how difficult it is for a DNA to unpack and replicate, as well as help estimate the properties of the enzymes that reorder the DNA. Furthermore, recent research has shown that the knot type of the DNA molecule has an effect on the function of the DNA, biologists theorize that using knot theory techniques it may be possible to bring further insight into the structure and manipulation of a DNA molecule.

3.2. **Optical Vortex Knots.** Biology is not the only science that has benefited from the study of knot theory; Physicists have put knot theory into practice in some astounding method. One such study was on optical vortex knots where researchers succeeded in designing a experiment which creates optical beams with isolated optical vortex loops in the form knots and links. In layman's terms, they tied light into knots. The lead Professor on the project, Professor Miles Padgett said: "The sophisticated hologram design required for the experimental demonstration of the knotted light shows advanced optical control, which undoubtedly can be used in future laser devices". These advancement in laser technology has the possibility to affect a variety of sectors, from beauty and healthcare to manufacturing and technology.

3.3. **Cryptography.** Another application of Knot theory is in programming, specifically encryption. In an increasing online world, cyber security is an ever-present matter. Encryption is key to ensuring your private messages stay private. The RSA algorithm is how modern computer encrypt and decrypt messages to ensure secure communication. The RSA works by producing two prime number product and an additional number as the public key. For example, consider a scenario where person 1 send a message to 2. 2 sends the public key which anyone can see, to 1. 1 uses the key to encrypt the message. 1 sends the encrypted message to 2, but none can decrypt it until 2 receives the message and, knowing the prime numbers, is able to decrypt it. While effective the RSA is not efficient. It's computationally hard and very costly in term of resources and time.

With such an extensive use, many people are considering ways to improve the RSA. One such development is from the of a knot-based encryption system. It would work in a similar way as the RSA. A & B share a finite list of prime knots. The message Is built up from a finite sequence of knots. Through a standard RSA protocol, B sends to A an ordered sub list of N prime knots. These composite knots are now sent to B. At this stage everyone has access to these strings of relative integers. B receives the composite knots. Since B knows the relevant prime knots, they can decrypt the composite knots, and thus decrypt the message. This method is more efficient then the orginal RSA system, however this method is still in development and more work is required to ensure the knot-based system can withstand a diverse range of attacks.

Even in its infancy the study of distinguishing, untangling and tangling knots has had a myriad of scientific uses, and this is still while these questions are still being figured out in full. One can only imagine all the possibilities that further research into mathematical knots may lead too.

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A BRIEF SUMMARY OF KNOT THEORY

OISÍN FIUZA, DANIEL GALLAGHER, AND TONY O'KELLY LYNCH

1. Introduction: What is a Knot?

In layman's terms, a knot is a simple object consisting of a simple string, such that when you "tangle" the string in some fashion, or simply leave it as it is, you join the two ends together to create a knotted loop[1]. There are, however, some key differences between a mathematical knot and a physical knot in a string: The string itself has no thickness. It merely exists as a line. As such, its cross section consists of a single point. It is also important to note that the knot does not intersect with itself at any point. Another important property of knots is that two are not considered different if one can deform the not in such a way that it creates another shape (so long as such deformations do not allow the curve to pas through itself). A deformation consists of an action that cause the "string" to curve, stretch, or distort, without breaking or tearing the string, as pictured below [1].

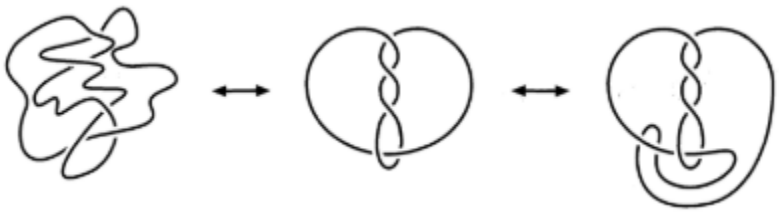

Figure 1. Same knot, different deformation

The most basic of knots, referred to as the unknot or the trivial knot, can be visualised as a simple string such that the two ends are connected together, forming a circle (or any deformations of a circle, such that the characteristics stated above are maintained). The next distinct knot considered to be the next simplest is referred to as the trefoil knot, pictured in 2[1]:

One important question that concerns knot theory is how do we distinguish between two knots, let alone deformations of different knots? This is one of the main reasons why distinguishing knots can be so difficult, even for the most basic of knots. One way that we do this is through knot invariants, i.e., characteristics of a knot which stay constant no matter how they are deformed (more on this later).

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FIGURE 2. (a) unkot (b) trefoil

2. Origins

According to Robert Osserman in Encyclopaedia Britannica [6], Carl Friedrich Gauss first took steps towards the theory of knot theory in 1800, but it was not until 1869 that modern knot theory was first considered. William Thomson (Lord Kelvin), a Scottish mathematician suggested that atoms might consist of knotted vortex tubes of the ether. This sparked the first attempt to give a formal definition regarding what classified a knot. This was done by Scottish mathematician-physicist Peter Guthrie Tait. Kelvin and Tait believed that they could explain why atoms absorb and emit light at the wave-length they do via an understanding and classification of all possible knots. Tait began listing unique knots which are now known as Tait conjectures on alternating knots. These conjectures were proved in the 1990s and were subsequently improved upon by C. N. Little and Thomas Kirkman.

It was later proven that Kelvins theory was not correct, but knots were continued to be studied as a purely mathematical theory for about 100 years. In 1984, New Zealand mathematician Vaughan Jones introduced a new concept called Jones polynomials, a knew knot invariant [3]. This breakthrough led to a discovery between knot theory and quantum field theory by Edward Witten, an American mathematical physicist. Both men were awarded Fields Medals for their work in 1990. Another American, William Thurston found a connection between knot theory and hyperbolic geometry. Since then, connection to knot theory have been found in biology, chemistry, and mathematical physics. Wikipedia states that the development of topology in the early 20th century spearheaded by Henri Poincaré, knot theory became relevant again and were investigated in this context. Max Dehn developed Dehn surgery, which related knots to the general theory of 3-manifolds, and formulated the Dehn problems in group theory, such as the word problem. In the first half of the 20th century, knot theory primarily consisted of study into the knot group and homological invariants of the knot complement.

Daniel S. Silver publication on Knot theory's origin contains some interesting facts, outlined in brief as follows [2]. It is likely that Gauss was motivated by the problem of determining the smallest region of the celestial sphere, the zodiacus, onto which the orbits of two heavenly bodies can be projected. Gauss discussed knots with his doctoral student, Johann Benedict Listing (1808-1882). Listing later coined the word "topology," a combination of the two Greek words topos (form) and logos (reason), to describe the new geometry of position. (He is responsible for many other scientific words and phrases commonly employed, "nodal points," "telescopic system" and "micron" being just a few.) Knots appear in Listing's first

monograph, appropriately entitled Topology (1847), but there is a lack of significant mathematical theorems about them. Listing had no effective tools at his command.

3. Goals of Knot Theory

Essentially, the main goal of knot theory is to distinguish between two or more knots [5]. Two knots may look very different at a given time, but they can be twisted and flipped into twins of each other, without making a cut or tear (which is cheating). Mario Livio [4] describes knot one of the main goals of knot theory has always been to identify properties that truly distinguish knots—to find what are known as knot invariants. A knot invariant acts very much like a "fingerprint" of the knot; it does not change by superficial deformations of the knot. Two major breakthroughs in knot theory occurred in 1928 and in 1984. In 1928, the American mathematician James Waddell Alexander discovered an algebraic expression (known as the Alexander polynomial) that uses the arrangement of crossings to label the knot. For example, the Alexander polynomial for the trefoil knot is $t2-t+1$, such that represents the invariant. Two knots that have different Alexander polynomials are indeed different; for example, the Alexander polynomial for the figure eight knot is $t2 - t3 + 1$. Unfortunately, two knots that have the same Alexander polynomial may still be different. Consequently, while it was certainly very useful, the Alexander polynomial was still not perfect for classifying knots. Decades of work in the theory of knots finally produced the second breakthrough in 1984. The New Zealander-American mathematician Vaughan Jones detected an unexpected relation between knots and another abstract branch of mathematics (known as von Neumann algebras). This led to the discovery of a more sensitive invariant than the Alexander polynomial, which became known as the Jones polynomial. The Jones polynomial distinguishes, for instance, even between knots and their mirror images, for which the Alexander polynomials were identical.

There are applications for knot theory in molecular biology, cryptography, string theory and many more. Bill Menasco, a knot theorist of 35 years, when asked about the real world relevance of knots, provide the following answer. "First, when cells divide, the DNA inside them must be replicated. This requires the DNA's double-helix structure to unfurl, generate a second copy of itself that's intertwined with the first, and then get untangled. To achieve this, enzymes help to cut, unknot and reconnect the strands. Knot theory provides insight into how hard it is to unknot and reknot various types of DNA, shedding light on how much time it takes the enzymes to do their jobs. In the realm of security, Menasco and his graduate students are partnering with the firm SecureRF to further develop the Algebraic Eraser, a security protocol that uses knots in cryptography. The protocol weaves mathematical information from knots into encryption keys that allow users to decipher secret data encoded in security badges, mobile payment devices and more."

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Abstract. The following paper is a brief introduction into a subsection of Topology known as knot theory. Knot theory primarily deals with identifying and distinguishing various knots through invariants present in the deformations of a knot. This paper aims to provide a brief introduction to those who are unfamiliar with knot theory to the concepts of knots, their origins in mathematical research, and the main goals of those who study this subject.

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KNOT THEORY

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(Communicated by Dorothy Buck)

Abstract. One of the main areas of interest for algebraic topologists is the study of mathematical knots, known as knot theory. This paper seeks to provide an introduction to knot theory, exploring what it entails, alongside a brief exploration into some of its history, its central concerns, and a primary unresolved challenge within this field of mathematical study.

Knots may be seen in many areas of life. Patterns of knots are visible throughout ancient history, including centuries ago in Irish history. Knot patterns were highly popular in Celtic artwork, for example in The Book of Kells, which exhibits beautiful, intricate knotwork. Examples of knots also remain hugely common in modern day-to-day living, such as in fishermen's ropes, a pair of entangled earphones, or the daily tying of one's shoelaces. However, those knots that we see in our day-to-day lives are not the same as the knots studied day-to-day by topological mathematicians. In contrast to the examples in everyday life, the knots of interest for mathematicians have connected ends. A mathematical knot can be thought of as being constructed through some interlacing entanglement or looping of a single string whose ends are then fused together. In mathematical terms, knot theory is "the study of closed curves in three dimensions, and their possible deformations without one part cutting through another"[1].

At a basic level, knots may be defined by their positions on the 'knot table,' wherein different knots are grouped together if they share the same number of crossings (i.e. points where a knot crosses over itself when projected onto a plane). Below are examples of the first three knots, with lowest possible crossing numbers.

The Unknot: Though its name seems to suggest it is not a knot at all, the simplest form of a mathematical knot is simply a circular loop, known as the 'unknot.' In its most basic, minimal form, this trivial knot has zero crossings, resembling a rubber band. No matter what ways it may be twisted or tied to cross over itself, it can always be un-twisted and un-tied and will retain its base form of a single loop. Given its lack of crossings, this knot is not actually considered to be a 'true' knot.

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FIGURE 1. The Unknot.

The Trefoil Knot: As there exist no knots with a minimum of one or of two crossings, the next type of knot which appears in a knot table is the trefoil knot, the knot with three minimum crossings. Again, as with the unknot, the trefoil can be twisted, deformed, and tied around itself to generate more crossings in appearance, but at its most unravelled state it will always have exactly three crossings. Technically, it can be considered that there exist two knots with three minimum crossings, however these are simply mirror images of the same trefoil knot.

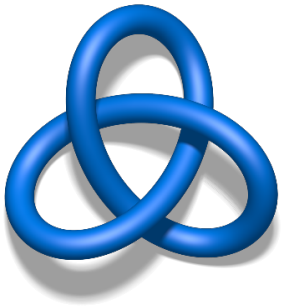
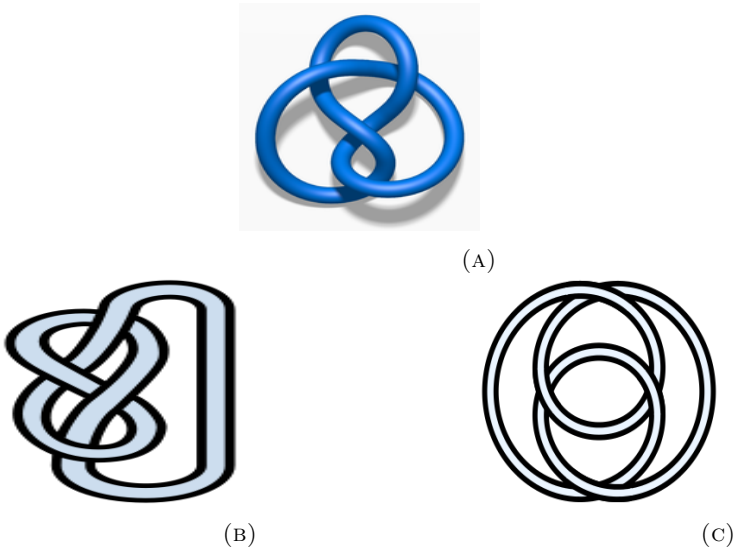


FIGURE 2. The Trefoil Knot.

The Figure Eight Knot: Similar to the first two examples, there is only one knot in existence with four minimum crossings, known as the figure eight knot. Even when retaining its minimal form, the figure eight knot can look different depending on how it is twisted, as visible in the images below. Note how, regardless of how it is projected, its crossing number is always exactly four.



(A) (B) (C)

FIGURE 3. The Figure Eight Knot in varying forms.

Knots pictured in this way, in their simplest form, are called prime knots. There is a limited number of prime knots in every grouping of knots with n crossings. However, as n increases, the size of the group of possible knots (each with a distinct shape due to the varying ways in which they interlace with themselves) which corresponds to this crossing number quickly becomes very large. For example, the number of different knots with crossing number five is two, and there are three different knots with crossing number six, but, skipping forward just a few groups, it has been shown that the number of different knots with crossing number ten is one hundred and sixty-five!

A compound knot is classified as a knot which is a merging of other knots together. Imagine, for example, cutting two separate trefoil knots so that each knot has two loose ends. If one then merges the ends of one knot to the ends of the other, together, they will create a compound knot with minimum crossing number six. (Note: compound knots are not included in the knot table. So, this does not correspond to any of the three knots with crossing number six.)

Topologically speaking, all of these knots are homeomorphic to a circle. Thus, to a topologist, the interesting thing about a knot is the way in which that circle is embedded in R3. In the 1920's, mathematician Kurt Reidemeister began using knot diagrams to come up with his theorem which shows that any two projections of isotopic (i.e. equivalent) knots are related to each other by combining transformations of the planar diagram. In other words, "two knot diagrams represent the same knot if and only if one diagram can be obtained from the other by a sequence of the moves" [2]. There are three such transformations, known today as Reidemeister moves, which are the following manipulations of a knotted string:

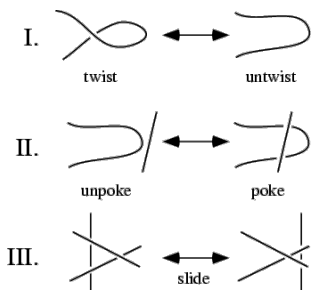


FIGURE 4. The Three Reidemeister Moves.

Some history of Knot Theory and The Recognition Problem: One of the principles challenges in knot theory, known as the recognition problem, is how to how to tell if two given knots are isotopic. When encountering a complicated knot, it can be essentially impossible to classify it by observation and determine what its minimum crossing number is. Even after playing around with it and manipulating it via Reidemeister's moves, it is very difficult to be certain that there aren't any additional moves to be made on the knot to render it into an even simpler form than what you have – a true 'Gordian knot' challenge.

Over the years, multiple mathematicians have come up with varying theories and methods which help to distinguish between different topological knots. These invariants, as they are known, actually date back to a familiar name for undergraduate mathematics students – Carl Friedrich Gauss. Gauss developed a way of counting up the 'linking' number of an intertwined combination of knots by orienting a knot in a certain direction and assigning +/-1 to each crossing the knot has, depending on if the knot crossing above another does so in the left or right direction, respectively, then counting up this number [3]. Later in the 1980's a theorem was developed which showed that a reduced, alternating diagram of a knot exhibits the minimal number of crossings, where reduced indicates that there is no crossing that can be untwisted without creating any additional crossing(s) elsewhere in the knot.

Other key breakthroughs in knot theory include the discoveries of the Alexander polynomial in the early 1900's, Belfast-born physicist Sir William Thomson's idea theorised in 1869 that atoms may be knots, Peter Tait's work (influenced by Thomson) towards introducing a classification for knots with ten or fewer crossings, and the Jones polynomial discovered years later in 1984. A range of other discoveries have been made between and since these breakthroughs, with many important contributions made across different fields of mathematical study. More recently, knot theory has even been recognised for its usefulness in other areas of study such as in understanding DNA.

Unfortunately, however, nobody has yet discovered a method to definitively tell whether two given knots are isotopic. Rather, this remains to be an open challenge for future topologists to work towards, and, hopefully, to someday solve.

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AN INTRODUCTION TO KNOT THEORY

SEÁN HESSION

ABSTRACT. The subject of Knot Theory dates back to 18^{th} Century. Although the original reasons for study in this field are now invalid, many other real world applications have been found. The fact that knot theory's foundations lay on concrete ideas mean that is accessible for those with little formal mathematical knowledge. Many proofs have been written by those with careers in other fields, even outside of the scientific community. This ease of access has made knot theory an exciting field of research in modern times.

1. INTRODUCTION

A knot is a closed curve in R^3, $K : [0, 1]$, that is continuous and 'nearly' injective; that is, it is injective at all points except for $K(0)$ and $K(1)$ with $K(0) = K(1)$. As in Topology, two knots, K_1 and K_2, are considered equivalent if K_1 can be made into K_2 under continuous deformations. The simplest knot, called the unknot or the trivial knot, is just a circle. The next simplest knot is the trefoil knot.

(a) The unknot. (b) A trefoil knot. [1]

Much of knot theory deals with distinguishing knots. For example, are the two knots mentioned above actually distinct? Another issue, if a knot is deformed beyond recognition, can it still be identified? This issue leads to the search of knot invariants, expressions associated with each knot and is unchanged by continuous deformations.

2. HISTORY

In the late 19^{th} Century, Lord Kelvin believed in the existence of the ether, a substance that occupied all space. He theorised that atoms were made up of knots of the ether. Under this assumption, Kelvin did much research and laid down the foundations of the theory. Unfortunately, Lord Kelvin's theory was disproven, but

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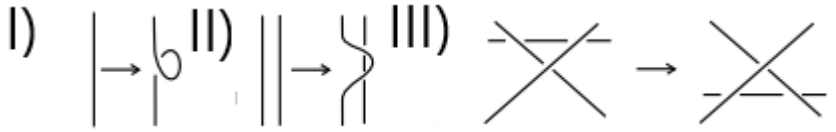
the mathematics communities' interest had been piqued. Others began research in the field and it wasn't long until the first table of knots was created.

3. In-depth Study

In order to study knots without the need for abstraction, knot diagrams were developed. Knot diagrams are projections of a knot onto a 2D plane. In these diagrams, any point where the knot intersects itself is called a 'crossing'. At each crossing, the broken line segment is understood to be crossing under the solid line segment. An alternate knot is one where each crossing alternates between crossing over and crossing under at each interval. With these conventions, knots can be drawn in two dimensions without ambiguity.

3.1. **Deformations.** In knot theory there are two types of deformations such that the end product is equivalent to the original knot. The usual stretching and shrinking is referred to as planar isotopy. The second is a set known as the Reidemeister moves. These moves state what is 'allowed' to be preformed on a knot without losing equivalency.



[2]

Type I moves deal with twists in a knot. Executing a type I moves means taking a segment of the knot without any crossings and twisting it to form a loop. The inverse of this is also permitted. A Type I move adds or removes a single crossing from a knot.

Type II moves add or remove two crossings that aren't alternate. A type II move takes two segments of the knot that share no crossing. Two crossings are forms by placing an arc from one segment exclusively under or over the other. Removing two crossings in a similar way is also permitted.

Type III moves do not change the number of crossings in a knot. A Type III move occurs when a segment of the knot is slid past a crossing.

Any and all of these deformations can be used on a knot in any order, any number of times but the end product will always be considered equivalent to the original knot. Passing sections of the knot through itself is not considered. For example, changing a segment of the knot from crossing under into crossing over is not considered and the end product is a different knot.

Deformation of a part of a knot to a single point is also not permitted. If the above was considered, it could be proven that every knot is equivalent to the unknot.

3.2. **Composition.** Composition of knots is analogous to multiplication in algebra. To form the composition of two knots, cut the two knots and fuse these endpoints together to form a new knot. The points to cut on each knot must be chosen so that when they are fused together, no new crossings are formed. It can be clearly seen that any knot composite with the unknot will produce the first knot. In the multiplication analogy, the unknot can be considered as 1.

This leads us to looking at previously known knots and trying to find compositions that form them. If a knot is not the composition of any two other non-trivial knots, it is said to be a prime knot. Therefore, the only composition that makes a

prime knot is itself and the unknot. When knots are tabulated, only prime knots are listed.

Where composition differs from multiplication is in the fact that two knots can form two different compositions based on where the knot is cut. In order to avoid confusion in this matter, knots can be oriented. To place an orientation on a knot, arrows are drawn to show clearly the direction of travel around the knot.

When the composition of two orientated knots is formed, if the orientation of the two knots is the same, the composite knot will always be equivalent regardless of where the cuts are made. Likewise, the composite of knots with differing orientations will always be equivalent. There is nothing preventing these two knots being equivalent under certain circumstances, but at most two unique knots can be formed.

An orientated knot is considered invertible if the knot can be deformed back to itself with an opposite orientation. Although it can be easy to prove a knot is invertible, it can be difficult to prove a knot cannot be inverted. Nobody has come up with a general method for proving if a knot is invertible.

3.3. **Links.** The study of knots naturally extends to studying links. Links are sets of knots tangled together. Links are made up of a number of unconnected components. A knot is simply a link of one component. Links are splittable if they can be deformed into unconnected components, separated by a plane.

It is useful to know just how connected links are. To do this a term known as the linking number is defined. It can be used to help distinguish links. To calculate the linking number, an orientation is chosen on each component of the link. At each crossing of different components, +1 is assigned if it is a type a) crossing, or -1 for a type b) crossing. The sum of these numbers is divided by two to give the linking number. The orientation of the links does effect the sign of the linking number, so the absolute value is usually used.

[1]

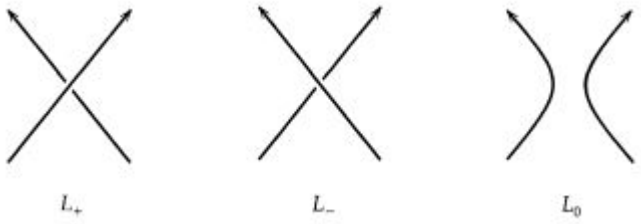
It can be clearly seen that Reidemeister moves do not effect the linking number. From this fact, it follows that the linking number is independent of the projection of the link. It can be said that the linking number is an invariant of an orientated link. Although the linking number is invariant, it is not unique. This leads to looking for other invariants and ways to distinguish links.

3.4. **Invariants.** As discussed earlier, an invariant is an expression to describe a knot, or link, that is independent of its projection. A lot of research has gone into this area. If a knot invariant could be found that is also distinct for each knot, it would make it possible to easily identify a knot. This is important because, as the number of crossing increase, it appears that the number of unique knots increase exponentially. There are 1,701,936 knots with 16 or fewer crossings. It then becomes no easy task to determine whether these knots are equivalent.

The concept of tricolouration is an invariant. A knot is said to be tricolourable if it can be coloured using at most three colours under the following conditions: At every crossing, all three colours, or else only one colour, meet; at least two colours must be used. It can be shown that Reidemesiter moves preserve tricolouration, making it an invariant. Although tricolouration can be used to dintinguish knots, it is not a unique identifier.

The unknotting number is another invariant. This is the lowest number of crossing that have to be changed in order to make the knot into the unknot.

The Jones' Polynomial, $V(L)$ is one of the most useful invariants to date. It is easy to calculate and, although it is not unique , it helps to distinguish many knots that were considered equivalent until its implementation. The Jones' polynomial is calculated by choosing a crossing L_+ and splitting it into L_- and L_0 as below.



[1]

The Jones' polynomial of the unknot is defined as 1. By using a recursion until the unknot is found, the Jones' polynomial is calculated using the equation,

$$t^{-1}V(L_+) = tV(L_-) + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})V(L_0)$$

By calculating these invariants, it can be determined that two knots are distinct if any one of them differ.

3.5. **Modern Applications.** After the existence of the ether was disproven, knot theory lost its applications. Ironically over 100 years later, applications were again found for knot theory in Chemistry.

Molecule made up of identical atoms can have different properties. To find out the cause of these differences lead to molecular diagrams. It was found that these molecules form knots and links. Chemists could then look to knot theory to help explain this different properties.

Knot theory can also be applied to the study of DNA. Enzymes acting on DNA strands cause different effects to take place. These can be made to resemble effects taking place on a knot.

4. Beyond This Paper

Of course there are many areas of knot theory that haven't been touched upon in this paper. Knots can exist in higher-dimensions and are not restricted to 1D curves in R^3. A great deal of effort goes into tabulating knots and proving that they are in fact prime and unique. There are many open questions in the field that have yet to be rigorously proven. For example, the Jones' polynomial for the unknot is 1 but it has not been proven that there exist no other knot with the same Jones' polynomial.

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Winners of the Field Medal Award in the Area of Topology

Ellen Hynes[1,*], Stephanie Moffatt[1,1]

Abstract

In this article we will be focusing on the history of the Fields Medal and some of the topologists who received these medals.

1. History of the Field Medal

The Field Medal is a medal awarded to three or four young mathematicians for outstanding contributions to the field of mathematics. The event takes place every four years was most recently held in Rio deJanero in 2018.
The Medal is named after Canadian mathematician James Charles Fields who was instrumental in the establishment of the award and design of the medal.
The medal was first awarded in 1936 to mathematicians Lars Ahlfors and Jesse Douglas.

2. Field Medal Recipients

2.1. Simon Donaldson

Donaldson is an English Mathematician who won the award in 1986, held in Berkely California. Donaldson attended Pembroke College in 1979 where he completed his degree in Mathematics. He later attended Worcecster College where he completed his postgraduate studies.
Here was where he specialized in topological spaces. Whilst still in College he published his paper on self-dual connections and the topology of smooth 4-manifolds with which he established and set himself apart in the world of Mathematics.

Theorem 1 (Donaldson's Theorem)**.** *A definite intersection form of a compact, oriented, smooth manifold of dimension 4 is diagonalisable.If the intersection form is positive definite, it can be diagonalized to the identity matrix over the integers.*

Proof of this theorem is based on moduli space of solutions to the anti-self-duality equations on a principal .
He then finds the dimension of the moduli space. He observed that the singular points in the interior of the moduli space looked like cones. Thus it is possible to compactify the moduli space by cutting off the cones and attach a copy of the complex projective plane and a copy of X itself at infinity. This shows that the intersection of X is diagonisable.
Combining The the Donaldson theory along with Michael Freedmans findings gives us:

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1) Any non-diagonalizable intersection form gives rise to a four-dimensional topological manifold with no differentiable structure (so cannot be smoothed).
2) Two smooth simply-connected 4-manifolds are homeomorphic, if and only if, their intersection forms have the same rank, signature, and parity

Figure 1: Simon Donaldson

2.2. Vladimir Voevodsky

Vladimir Voevodsky is a Russian American mathematician.He won the Field Medal in 2002 for development of homotopy theory for algebraic varieties as well as formulating motivic cohomology.He completed his PHD of mathematics in 1992 despite not having received a diploma from Moscow State University after being forced to leave due to poor attendance among other reasons.

Homeotopy Theory for algebraic varieties:
Two paths with common endpoints are called homotopic if one can be continuously deformed into the other leaving the end points fixed and remaining within its defined region.
Vladimir came up with Motivic homotopy theory that blends together both algebra and topology. He is also known for the proof of the Milnor conjecture and motivic Bloch–Kato conjectures both of which have revolutionalised current mathematics.

Figure 2: Vladimir Voeodsky

2.3. Vaughan F.R Jones

Sir Vaughan Frederick Randal Jones, one of the world's mathematicians, is a New Zealand mathematician, known for his work on Von Neumann algebras and knot polynomials. He was awarded a Fields Medal in 1990, and famously wore a New Zealand rugby jersey when he gave his acceptance speech in Kyoto.

He was born on the 31 December 1952 in Gisborne, New Zealand and was brought up in Cambridge, New Zealand. His undergraduate studies were at the University of Aucklan, from where he obtained a BSc in 1972 and a MSc in 1973. For his graduate studies he went to Switzerland, where he completed his PhD at the University of Geneva in 1979. In 1980, he moved to the United States where he taught at University of California, Los Angeles and then the University of Pennsylvania before being appointed as Professor of Mathematics at the University of California, Berkeley, where he remains as Professor Emeritus. Since 2011, he has also been at Vanderbilt University as Stevenson Distinguished Professor of Mathematics.

Jones won the Field Medal for "his discovery of an unexpected link between the mathematical study of knots- a field that dates back to the 19th century- and statistical mechanics, a form of mathematics used to study complex systems with large numbers of components." Not only did his discovery, known as the Jones polynomial, come as a complete surprise when it was announced, but also its importance has increased over time as mathematicians have discovered that it has connections with a number of widely separated areas of mathematics and physics. His work is also being used by molecular biologists studying DNA. It turns out that the mechanisms used by cells to untangle the double-helix strands of DNA during replication and recombination are strikingly similar to the mathematical moves used to generate Jones polynomial.

Jones Polynomial:
The Jones polynomial of an oriented link L is

$$V(L) = [(-A)^{-3w(D)} < D >]_{t^{1/2}=A^{-2}}$$

where D is any diagram of L.

Figure 3: Vaughan Jones

2.4. Stephen Smale

Stephen Smale, born on July 15 1930, is an American mathematician, known for his research in topology, dynamical systems and mathematical economics. He was awarded the Field Medal in

1966 and spent more than three decades on the mathematics faculty of the University of California, Berkeley.
Smale was born in Flint, Michigan and entered the University of Michigan in 1948. After a few years of only mediocre grades, and once even threatened to be kicked out, Smale finally earned his PhD in 1957. Smale began his career as an instructor at the University of Chicago. In 1958, he astounded the mathematical world with a proof of a sphere eversion. He then cemented his reputation with a proof of the Poincaré conjecture for all dimensions greater than or equal to 5, published in 1961. In 1962, he generalised the ideas in a 107-page paper that established the h-cobordism theorem.
Smale won the Field Medal because of his "work in differential topology where he proved the generalised Poincaré conjecture in dimensions greater than or equal to 5. Every closed, n-dimensional manifold homotopy- equivalent to the n-dimensional sphere is homomorphic to it. Introduced the method of handle-bodies to solve this and related problems."

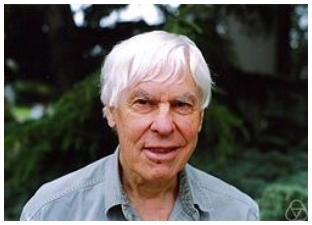

Figure 4: Stephen Smale

3. Conclusion

As we can see topology is an area that is integral in all parts of mathematics. Topology presents itself in everyday life and we have the above mathematicians to thank for their contributions to the study of topology.

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MANIFOLDS

DARRAGH MAC MANUS & AISLING MCSHARRY

Abstract. The primary focus of this paper will be to explain what a manifold is and how topology applies to manifolds. This topic does fall outside of the syllabus of the MA342 Topology course. However, many of the concepts we learnt in class are essential in understanding and defining manifolds. As well as this, we will also give some examples of manifolds and their properties.

1. Definition

Let's begin with the basic definition of a manifold.

Definition 1.1. A manifold is a topological space that locally resembles Euclidean space near each point.[3]

This definition contains some key concepts we must grasp first. We understand that a Euclidean space is the n-dimension space of real numbers, regularly the 2-dimensional and 3-dimensional spaces. For a topological space to be considered locally Euclidean, there must be real number n such that every point in the topological space has a neighbourhood which is homeomorphic to $\mathbb{R}^n$ [4]. Now that we have this basic definition, we can move onto a more precise version.

Definition 1.2. An n-dimensional manifold is a topological space in which each point has a neighborhood that is homeomorphic to the Euclidean space of dimension n.[3]

From the MA342 course, we learnt that a neighbourhood is a set of points surrounding a point it in which one can move a certain distance in any direction and still be contained in the set. Furthermore, we learnt that two topological spaces, X and Y, are homeomorphic if there exists some homeomorphism between them f:X→Y. A homeophormism is a continuous function between topological spaces which has a continuous inverse. In terms of topology, if two spaces are homeomorphic there are considered the same since they can be stretched and moulded into each other. Despite understanding these key concepts, it can be difficult to envision what a manifold actually is. To picture what a manifold looks like, we must use different dimensions.

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2. 1-manifolds

The most logical dimension to begin with is the lowest, which is 1. Let's consider a circle in 2-dimensions, as displayed in FIGURE 1. If we examine each arc of the circle, we can observe that it can be moulded into a 1-D line segment. This means that there exists a homeomorphism between the arc and the line. Let's consider taking the smallest possible arc. This minuscule segment of the circle will look like a straight line on close examination. More formally, "if you take an infinitesimal arc, it will "locally" resemble a one dimensional line segment" [2]. This concept can applied to any closed loop in 2-dimensions. This includes as circles, parabolas, hyperbolas, and cubic curves, all seen in FIGURE 2. Everyone of these shapes has a one-to-one correspondence to a 1-D line segment. This correspondence is considered a natural mapping. This mapping is defined as a homeomorphism and therefore all of these closed loops are classified as 1-manifolds.

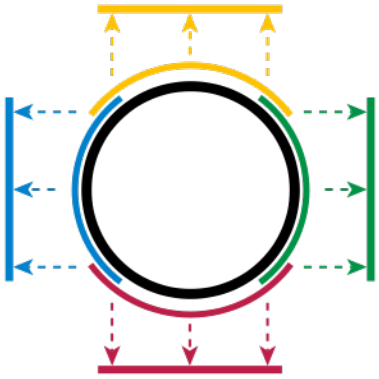

Figure 1. [2]

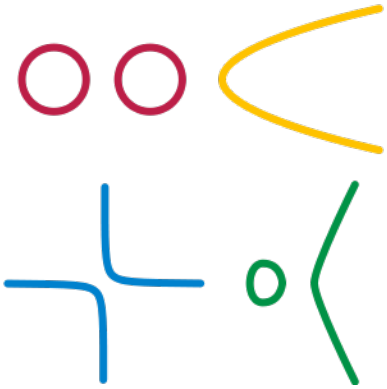

Figure 2. [2]

3. 2-Manifolds

Let's move onto 2-manifolds. Just as before, this term refers to a 3-dimensional shape which is homeomorphic to a 2-dimensional space. As an example, let's use the most basic 3-dimensional shape, the sphere. Consider taking a section of the sphere, as seen in FIGURE 3. If we create a infinitely small patch of the sphere, it will locally resembles a 2-D Euclidean plane. There exists a homeomorphism between this 3-D shape and the 2-D plane, and therefore the sphere is a 2-manifold. This homeomorphism exists for all non-intersecting closed surfaces in $\mathbb{R}^3$.[2] Examples of 2-manifolds include the sphere, the torus, the double torus, the Klein Bottle, the cross surface and the Mobius strip, all exhibited in FIGURE 4 and 5 [2][11].

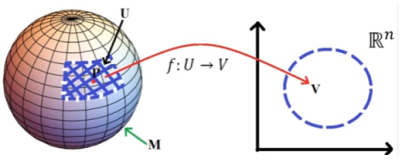

Figure 3. [1]

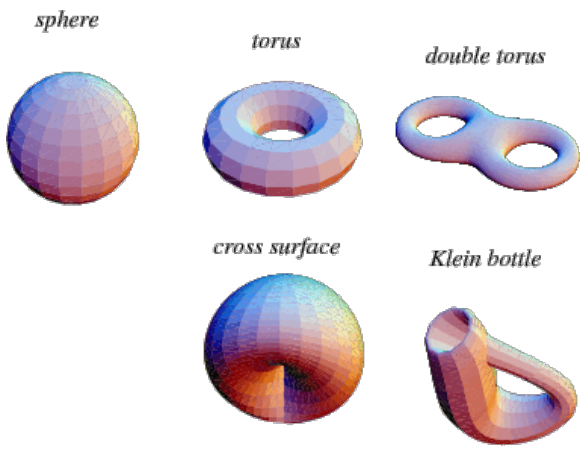

FIGURE 4. [2]

4. Properties of 2-manifolds

A key property of a manifold is whether it is orientable or non- orientable.

Definition 4.1. A surface is non-orientable if a two-dimensional figure can be continuously moved around the surface and back to it's starting point so that it looks like it's mirror image.[7]

A simpler explanation is that a manifold is said to be non-orientable if you can walk along some path and come back to where you started but reflected. A Mobius strip is an example of a non-orientable surface. As it is a one sided surface, you always return to where you started, but mirrored as seen in FIGURE 5. Conversely, an orientable surface is shape that fails the above definition. A torus is an orientable surface as you can continuously go around this shape without being reflected as shown in FIGURE 4.

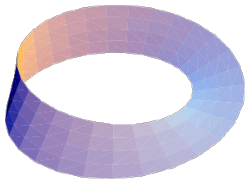

FIGURE 5. [11]

This leads me to a genus, another important property of a manifold.

Definition 4.2. The genus of an orientable surface is an integer representing the maximum number of cuttings along non-intersecting closed curves without the manifold disconnecting.[10]

Less formally the genus is simply the number of "holes" in a manifold. A manifold that has no "holes" would have a genus 0. With regards to the manifolds displayed in FIGURE 4, the sphere has a genus 0, a torus has a genus 1, a double torus has a genus 2, a cross surface has a genus 1 and a Klein bottle has a genus 2.

5. Euler Characteristics

The genus of a manifold is closely related to it's Euler characteristic.[10] Lets begin by giving a basic definition of the Euler characteristics.

Definition 5.1. The Euler characteristic χ of a manifold is χ=V-E+F where V, E and F denote the numbers of vertices, edges and faces in any triangulation of the surface, respectively.[6]

Euler characteristics were originally classically defined for the surfaces of a polyhedron, which is a 3-dimensional shape with polygonal faces, straight edges and sharp vertices.[6] The Euler characteristic of the manifolds in FIGURE 4 are listed below:

Sphere χ= 4V -6E +4F =2
Torus χ= 1V -2E +1F =0
Double Torus χ= 0V -4E +2F =-2
Klein Bottle χ= 1V -2E +1F =0

As mentioned previously, the genus and the Euler characteristic are closely related. For an orientable manifold the relationship is X= 2 – 2g, whereas for a non-orientable manifold the relationship is X= 2 -g.[10]

6. History

[8]

Georg Friedrich Bernhard Riemann was a German mathematician who was known for discovering important non-Euclidean geometries.[8] Riemann was not the first to study the concept of manifolds but significantly developed the theory.The name manifold comes from Riemann's original German term that directly translates into manifoldness.[8] He discovered that a space does not need to be an ordinary Euclidean space, that it could have any dimension and therefore, he was one of the first to generalise surfaces in higher dimensions. This great German mathematician even has a manifold named after him, the Riemann surface, which is characterised as a one-dimensional complex manifold.

Riemann had a close connection to another famous German mathematician, Carl Friedrich Gauss. In fact, Riemann was a student of Gauss'.[9] One of the most important developments that Gauss made was to consider abstract spaces as mathematical objects. During the 19th century, the Euler characteristic was linked with Gaussian Curvature and the product was the Gauss-Bonnet theorem, which states

[5]

Definition 6.1. The curvature of a surface can be determined entirely by measuring distances along paths on the surface.[9]

It was Riemann who extended this theorem to higher dimensional spaces which are now known as manifolds.[9]

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THE FIELDS MEDAL AND EXCELLENCE IN TOPOLOGY

MCCABE, TARA AND MORTELL, DENIS

Abstract. The Fields Medal is one of the most prestigious prizes that can be awarded for mathematics. While many have received the award since its inception in 1936, some recipients have been more memorable than others. In the field of topology, three of the most remarkable recipients are Caucher Birkar, Maryam Mirzakhani and Grigori Perelman. These three topologists all contributed hugely to their fields and also have an exceptional story to accompany their success.

1. Introduction to the Fields Medal

Prestige in mathematics has many faces. There are numerous awards for excellence in various areas of mathematics, some of which are equivalent to the Nobel Prize. There is the Abel Prize, awarded by the King of Norway to those who excel in their respective field of study, the Wolf Prize in Mathematics, one of six prizes awarded by the Wolf Foundation of Israel, and the Chern Medal, a newer award which recognizes lifetime achievement in mathematics. Finally, there is the Fields Medal. The Fields Medal is an award that values outstanding contributions in mathematics credited to young scientists, with the hope of encouraging a lifetime of work. The award is given to 2 to 4 mathematicians under the age of 40 and is awarded at the International Congress for the International Mathematical Union. This is an event which only happens every 4 years. The Medal was first awarded in 1936 to both Lars Ahlfors and Jesse Douglas from Finland and the U.S. respectively. Ahlfors worked with surfaces "related to Riemann surfaces of inverse functions of entire and meromorphic functions", and Douglas worked on the Plateau problem, concerned with finding minimal surfaces connecting and determined by some fixed boundary. While the Fields Medal is known as the "Nobel Prize of Mathematics", they are different in two main ways: the Fields Medal has an age limit and is only awarded every four years. As well as these differences, a recipient can only be awarded one medal. There is a cash prize given to those who win the Fields Medal, which amounts to CA$15,000. The most recent award ceremony was on August 1st 2018, in Rio de Janeiro, Brazil.

2. Caucher Birkar

Caucher Birkar is one of the more recent winners of the Fields Medal. More remarkably, he is the only person to have been presented two medals. This strange and seemingly improper occurrence, however, was not because he had made two great strides in mathematics. The medal he had won in 2018 was stolen only thirty

2010 *Mathematics Subject Classification.* Primary 55-01.

minutes after the event and he was presented his second medal a few days later. Birkar is originally from Iran and is affiliated with Cambridge university. He was awarded the Medal "for the proof of the boundedness of Fano varieties and for contributions to the minimal model program" in the field of algebraic geometry.

A Fano Variety, introduced by Gino Fano, is a complete variety X (smooth over a field), whose anticanonical bundle Kx is ample. The minimal model program has also led to the studies of these Fano varieties with different types of singularities. The Minimal Model Program gives a way to identify special varieties in each class. They are in some ways the simplest, but once found, they can be used to construct other more complicated varieties. In 2016, Birkar presented two papers which focus on the Fano varieties. The main focus point is Birkar's proof of Borisov-Alexeev conjecture. This conjecture stated that Fano varieties form a bounded family. Birkar also showed that once in a fixed dimension theses Fano varieties can be indexed using a finite number of singularities that arise when using the Minimal Model Program. These steps forward in Mathematics by Birkar are expected to lead to the solution of many problems in birational geometry. Birkar is a very remarkable winner of the Fields Medal, not just because of the extraordinary circumstances of his award, but because of his exemplary work in mathematics.

3. Maryam Mirzakhani

Maryam Mirzakhani was the first woman to win the Fields Medal in 2014. Originally from Iran, Mirzakhani was affiliated with Stanford University when she was awarded the Medal. She won the award "for her outstanding contributions to the dynamics and geometry of Riemann surfaces and their moduli spaces". Mirzakhani was born in 1977 in Tehran, Iran, and at the age of 27 she received her Ph.D. from Harvard University. Mirzakhani distinguished herself during her time in Harvard "by her determination and relentless questioning, despite the language barrier"[3]. In the years between 2004 and 2008, she was a Clay Mathematics Insititute Research Fellow, someone who has been selected for their research achievements and their potential to become a leader in research mathematics[4]. Other achievements for Mirzahkani include the 2009 Blumenthal Award and in the 2013 Satter Prize of the American Mathematical Society.

Mirzakhani began working with closed geodesics on hyperbolic surfaces, which is a surface with length that cannot be made shorter by deforming it. She used the prime number theorem for geodesics and questioned what happens when you only consider only simple closed geodesics, which are geodesics that do not intersect themselves. She noticed that in this case, the growth is no longer exponential and instead follows $\mathrm{L}^{6\mathrm{g}\text{-}\mathrm{g}}$, where g is the genus. She showed that the number of closed geodesics is asymptotic to $c{\cdot}L^{6g\text{-}g}$ for any large L (tending towards infinity) and c (the constant) is dependent on the hyperbolic structure. Mirzakhani proved it by considering all hyperbolic structures simultaneously. In Mirzakhani's proof, she encounters a symplectic structure. This structure allows you to measure volumes and not lengths. She calculates certain volumes in moduli space and then deduces that counting result for simple closed geodesics. Mirzakhani worked with moduli, which is a space that is completely inhomogenous. This means, that "every part looks totally different from every other part"[5]. She proved that the rigidity in homogenous spaces had an echo in the inhomogenous moduli space.

Since she became the first woman to win the Fields Medal in 2014, Maryam Mirzakhani has sadly passed away. She died on the 14th of July 2017 at the age of forty. Her work to this day has opened doors for future generations of mathematicians make new discoveries. She will always be remembered, not only as a phenomenal leader in the area of topology, but also as an icon for women, both inside and outside of mathematics, worldwide.

4. Grigori Perelman

A particularly interesting case of a topologist and the Fields Medal is one in which the Medal was refused by the recipient. Grigori Perelman was awarded the Fields Medal in 2002 for his proof of the Poincaré Conjecture, a notoriously difficult and complex conjecture formulated in 1904[1]. Henri Poincaré, considered to be one of the founders of topology, posed the infamous question offhandedly at the end of a sixty-five-page paper. Two-dimensional manifolds were well-understood by the mid-century, however the difficulty lay in clarifying whether that which was true in two dimensions would also be true in three. By the 1980s, Poincare's conjecture had been proven in all dimensions except the third. This conjecture went unproven for 98 years, with a great multitude of attempts from topologists all over the world, but to no avail. The problem was so difficult that in 2000, the Clay Mathematics Institute declared the Poincaré Conjecture "one of the seven most important outstanding problems in mathematics" and set a prize of one million dollars to whomever could prove it[2]. Finally in 2002, Grigori Perelman shared his proof online in three segments, as opposed to submitting it to be peer-reviewed. This demonstrated Perelman's certainty of its validity and also how unorthodox he was. Four years later, the proof was validated by at least two teams of experts and Perelman was credited with being the first person to provide a valid proof of the Poincaré Conjecture. He refused the Fields Medal as he believed that a reward was unnecessary and that "if the proof is correct then no other recognition is needed."[2] Perelman's unusual personality is reflected in his rejection of the Fields Medal and solidifies his place as one of the most memorable recipients.

5. Conclusion

Excellence in mathematics is celebrated all over the world in many different forms, and the Fields Medal is one of them. The Medal has no boundaries, and rewards outstanding work to all who deserve it, regardless of nationality, gender, or any other social barrier. It recognises the contributions of young scientists and encourages them to continue a lifetime of discovery and participation in the mathematical realm.

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PERSISTENT HOMOLOGY

BRIDGET MCGOWAN MURTAGH, HANNAH MCEVOY, AND BETH CONNERTY

(Communicated by Julie Bergner)

Abstract. There are two major flavours of Topological Data Analysis: Persistent homology and mapper. Both are very useful and can be used to complement each other. In this paper we cover persistent homology. This paper outlines what persistent homology is, how it is used to understand point cloud data sets and some applications of persistent homology in Topological Data Analysis.

1. Introduction

Topological Data Analysis (TDA) is a way of applying methods from topology to examine datasets. TDA is a relatively new branch of statistics and is still developing. When analyzing data, it is important to study the 'shape' of the data. Techniques from topology allow mathematical examination of 'shape'. The main tool used in TDA is persistent homology, a method of computing topological features of shapes and functions . Persistent homology was introduced for studying qualitative properties of large empirical data sets and is the flagship tool for TDA.

2. Persistent Homology

Homology is a general way of associating a sequence of algebraic objects with other mathematical objects, such as topological spaces.

Persistent Homology (PH) is an adaptation of homology to cloud point data. It is used to study qualitative features of data. Often in data analysis, algorithms used require the choice of particular parameter and it can be difficult to select the correct collection of parameters to study a data set. The benefit of using persistent homology is that the information obtained from all parameters can be utilised. Persistent homology let's us ask topological questions of our data in a reliable way without having to alter the data in any way. It gives us a way to find interesting patterns in the data without having to "downgrade" the data in any way. PH let's us leave our data in it's original, ultra-high dimensional space and tells us how many clusters there are, and how many loop-like structures there are, all without being able to actually see it.

To find the persistent homology of a space, the space must be represented as a simplicial complex. A distance function on the underlying space corresponds to a 'filtration' of the simplicial complex; that is a sequence of increasing subsets. There

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are various different software packages in languages such as MATLAB, Python and Java that are used to compute properties relating to these filtrations.

3. Point Cloud Datasets

Often, data is represented as an unordered sequence of points in a Euclidean n-dimensional space $\mathbb{E}^n$. (Ghrist 2007) The global 'shape' of the data may often provide important information about the underlying phenomena which the data represents. One type of data set for which global features are present and significant is the so-called *point cloud data* coming from physical objects in 3-D.

A *point cloud* is a set of data points in space (Wikepedia 2019). By treating the underlying dataset as a point cloud and analyzing its topology, researchers are able to draw important conclusions about the data set. These points represent the X, Y, and Z geometric coordinates of a single point on an underlying sampled surface (Gray 2019). Point clouds are a way of assembling a large number of single spacial measurements into a dataset that then represents a whole. Point clouds are most commonly generated using 3-D laser scanners and LIDAR (light detection and ranging) technology and techniques in which the distance of a ray of light from an object is measured as the ray travels on the objects surface. That way up to 750,000 datapoints per second can be recorded. Point clouds can also be synthetically generated from a computer program. A computer does not have to care about scale or rotation of points, only the position and maybe colour. When colour information is present, the point cloud becomes 4-D. Point clouds are generated and then used to measure physical spaces. For example, point clouds are used as a non-intrusive way to measure buildings like schools and hospitals. Point clouds can also be used to create 3-D CAD models for manufactured parts, for metrology and quality inspection and for a multitude of visualization, animation, and mass customization applications.

4. Representing the space as a simplicial complex

Simplicial complexes are the main data structure used to represent topological spaces in TDA. To find the persistent homology of a space, the space must first be represented as a simplicial complex When a dataset is sampled, the goal is to obtain information about the underlying phenomenon represented by the data. When the object of study is a 3-D object it is important to detect global features such as the geometric shape, number of components, loops and holes through the surface, or voids inside the object. For that purpose we sample points from a given object to obtain a point cloud dataset. In general, a point cloud can be sampled in an n-dimensional metric space. The sampled points (or their subset) represent vertices that mutually connect to form a structure called a simplicial complex or a stream. First we construct an abstract simplicial complex and then we assign coordinates to embed the complex into a metric space.

We start with the following definition (Violeta Kovacev-Nikolic 2012):

Affinely Independent Points: Let $x_0, x_1, ..., x_k$ be points in an n-dimensional Euclidean space $\mathbb{E}^n$; these points are affinely independent if and only if vectors $x_i – x_0$, $1 \leq i \leq k$, are linearly independent.

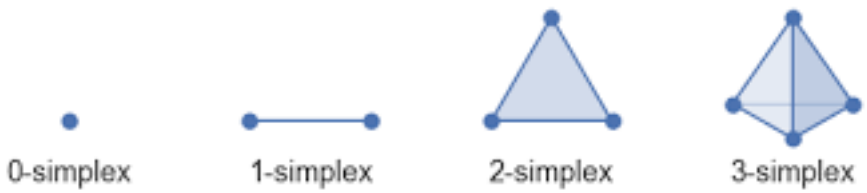
An n-dimensional space can have at most n linearly independent vectors so there can be at most n+1 affinely can independent points. Based on this, we define a k-simplex.

k-simplex: A k-simplex of k+1 affinely independent points $x_0, x_1, ..., x_k$ in an n-dimensional Euclidean space $\mathbb{E}^n$. is defined as the set of all linear combinations of the following form

$$\sigma(x_0, x_1, ..., x_k) \sum_{i=0}^{k} \lambda xi \tag{4.1}$$

where all λi are nonnegative and $\sum_{i=0}^{k} \lambda i = 1$

For k = 0, 1, 2, and 3, the corresponding k-simplex is just a point, a line segment, triangle, and tetrahedron, respectively as shown below.



Now we can define an abstract simplicial complex.

Abstract Simplicial Complex: Let σ be a simplex with its non-empty subset τ that we will call a face. Then an abstract simplicial complex K represents a finite collection of simplices such that it is closed under taking faces and has no improper intersections. More formally,

- $\sigma \in k$ and $\tau \leq \sigma$ implies $\tau \in k$
- $\sigma_1, \sigma_2 \in k$ implies $\sigma_1 \cap \sigma_2$ is either an empty set or a face of both

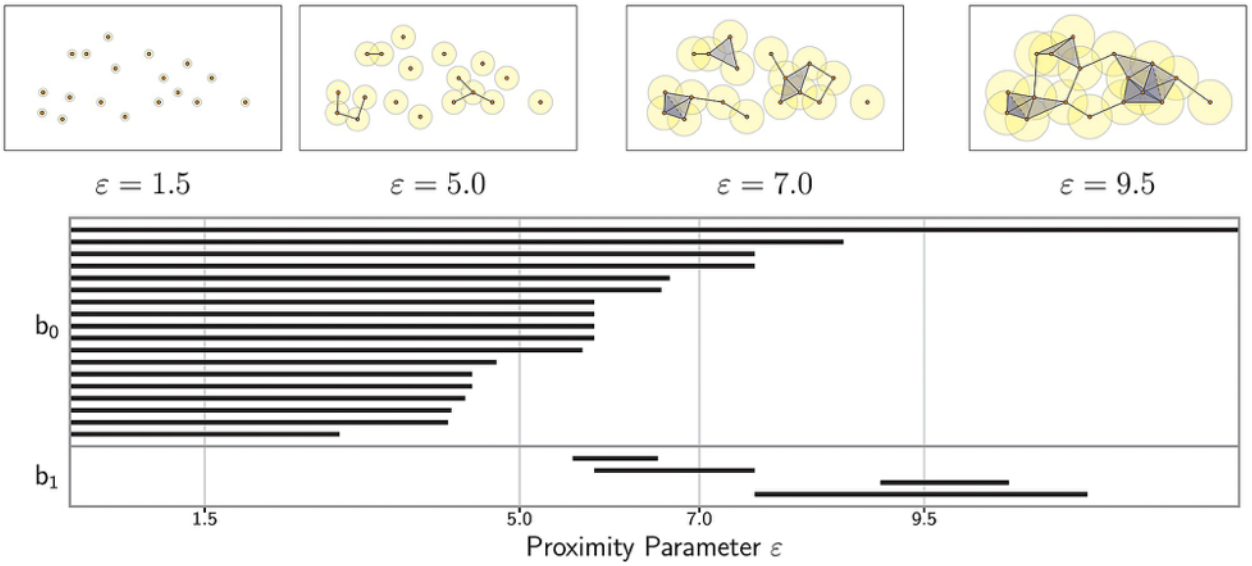
Given a simplicial complex, there is a corresponding abstract simplicial complex. A realization of an abstract simplicial complex is obtained by assigning actual coordinates to the vertices. We also define a rule, called filtration, which determines when an line segment (edge), triangle, or tetrahedron are formed. Such a filtered simplicial complex is called a stream. Although a filtration represents the distance at which two vertices bond, it is common to refer to this parameter as if it were time. Filtrations reveal more detailed structure in the complex, and provide tools for understanding how that structure arises.

5. Applications of Persistent Homology

Using the tool of persistent homology, TDA has been applied to a wide variety of real world problems, among which image compression, neuroscience, and shape or pattern recognition are just a few of many examples.

Network neuroscience is an approach to studying the human brain that conceptualizes the brain as a network of faces, edges and vertices. Persistent homology can be used to build a complete descriptor of this intricate network, by recording the evolution of a particular topological feature called a cavity as one moves through a network. A cavity is an innate topological property of neural systems that may be essential for system function. (Sizemore et al 2019)

The persistent homology of a point cloud is typically represented by barcodes or persistence diagrams. A barcode graph is shown below. This graph encodes all the topological features we're interested in in a compact and visual way. The idea behind this representation is the ability to qualitatively filter out topological noise and capture significant features.



Given barcodes or persistence diagrams, one can begin to classify different point clouds, or datasets, based on their topological features (McGuirl 2017). This method has been successfully applied to a range of fields such as image processing, evolutionary biology and shape segmentation and more. Types of data sets that can be studied with PH include finite metric spaces, digital images and networks.

In conclusion, persistent homology is a new mathematical concept that has received attention from inside and outside mathematics. It is a powerful tool and attention on it's applications is constantly growing.

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THE FIELDS MEDAL AND ADVANCEMENTS IN THE WORLD OF TOPOLOGY IN THE 20TH CENTURY

A. MC GUINNESS AND K. PAWELEC

(Communicated by David Futer)

Abstract. In this article we will discuss the importance of the Fields Medal in the world of mathematics and look at some of the most notable topologists who have earned this prestigious award since its conception over 80 years ago. In particular, we will be discussing the lives of past winners Sergei Novikov and Michael Freedman and their contributions to the world of topology in the 20th century.

1. Introduction

What is the Fields Medal? The Fields Medal is a prize awarded to two, three, or four mathematicians under 40 years of age at the International Congress of the International Mathematical Union, a meeting that takes place every four years. It is essentially the mathematician's Nobel Prize, although there are some key differences, such as frequency of awards and age restrictions. It is one of the highest honours a mathematician can be awarded. According to the annual Academic Excellence Survey by ARWU, the Fields Medal is consistently regarded as the top award in the field of mathematics worldwide, and in another reputation survey conducted by IREG in 2013–14, the Fields Medal came closely after the Abel Prize as the second most prestigious international award in mathematics. The prize comes with a monetary award which, since 2006, has been $15,000. The name of the award is in honour of Canadian mathematician John Charles Fields. Fields was instrumental in establishing the award, designing the medal itself, and funding the monetary component.

2. The life and works of Sergei Novikov

Sergei Novikov was born on 20 March 1938 in Gorky, Soviet Union (now Nizhny Novgorod, Russia). In 1955 Novikov entered Moscow State University. He grew up in a family of talented mathematicians. His father was Pyotr Sergeyevich Novikov, who gave the negative solution of the word problem for groups. His mother Lyudmila Vsevolodovna Keldysh and uncle Mstislav Vsevolodovich Keldysh were also important mathematicians. In 1955 Novikov entered Moscow State University and graduated in 1960. Four years later he received the Moscow Mathematical Society Award for young mathematicians. In the same year he defended a dissertation for the Candidate of Science in Physics and Mathematics degree at the Moscow State

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University, this is equivalent to a PhD. In 1965 he defended a dissertation for the Doctor of Science in Physics and Mathematics degree there. In 1966 he became a Corresponding member of the USSR Academy of Sciences.

2.1. **Research and contributions in topology.** Novikov's early work was in cobordism theory, in relative isolation. Among other advances he showed how the Adams spectral sequence, a powerful tool for proceeding from homology theory to the calculation of homotopy groups, could be adapted to the new (at that time) cohomology theory typified by cobordism and K-theory. This required the development of the idea of cohomology operations in the general setting, since the basis of the spectral sequence is the initial data of Ext functors taken with respect to a ring of such operations, generalising the Steenrod algebra. The resulting Adams–Novikov spectral sequence is now a basic tool in stable homotopy theory. This can be found in [1]. Novikov also carried out important research in geometric topology, being one of the pioneers with William Browder, Dennis Sullivan and Terry Wall of the surgery theory method for classifying high-dimensional manifolds. He proved the topological invariance of the rational Pontryagin classes, and posed the Novikov conjecture. This work was recognised by the award in 1970 of the Fields Medal. He was not allowed to travel to Nice to accept his medal, instead he received it in 1971 when the International Mathematical Union met in Moscow. From about 1971 he moved to work in the field of isospectral flows, with connections to the theory of theta functions. Novikov's conjecture about the Riemann–Schottky problem (characterizing principally polarized abelian varieties that are the Jacobian of some algebraic curve) stated, essentially, that this was the case if and only if the corresponding theta function provided a solution to the Kadomtsev–Petviashvili equation of soliton theory. This was proved by Shiota (1986), following earlier work by Arbarello and de Concini (1984), and by Mulase (1984)[2]

2.2. **Later career.** Since 1971 Novikov has worked at the Landau Institute for Theoretical Physics of the USSR Academy of Sciences. In 1981 he was elected a Full Member of the USSR Academy of Sciences (Russian Academy of Sciences since 1991). In 1982 Novikov was also appointed the Head of the Chair in Higher Geometry and Topology at the Moscow State University. In 1984 he was elected as a member of Serbian Academy of Sciences and Arts. As of 2004, Novikov is the Head of the Department of geometry and topology at the Steklov Mathematical Institute. He is also a Distinguished University Professor for the Institute for Physical Science and Technology, which is part of the University of Maryland College of Computer, Mathematical, and Natural Sciences at University of Maryland, College Park and is a Principal Researcher of the Landau Institute for Theoretical Physics in Moscow. In 2005 Novikov was awarded the Wolf Prize for his contributions to algebraic topology, differential topology and to mathematical physics. He is one of just eleven mathematicians who received both the Fields Medal and the Wolf Prize.

2.3. **The Novikov conjecture.** The Novikov conjecture concerns the homotopy invariance of certain polynomials in the Pontryagin classes of a manifold, arising from the fundamental group. According to the Novikov conjecture, the higher signatures, which are certain numerical invariants of smooth manifolds, are homotopy invariants. The conjecture has been proved for finitely generated abelian groups. It is not yet known whether the Novikov conjecture holds true for all groups. There are no known counterexamples to the conjecture.

2.3.1. *Precise formulation of the conjecture.* From this article [3]. Let G be a discrete group and BG its classifying space, which is a Eilenberg–MacLane space of type $K(G,1)$, and therefore unique up to homotopy equivalence as a CW complex. Let

$$f\colon M \to BG$$

be a continuous map from a closed oriented nn-dimensional manifold M to BG, and

$$x \in H^{n-4i}(BG;\mathbb{Q}).$$

Novikov considered the numerical expression, found by evaluating the cohomology class in top dimension against the fundamental class $[M]$, and known as a higher signature:

$$\langle f^*(x) \cup L_i(M), [M] \rangle \in \mathbb{Q}$$

where L_i is the i^{th} Hirzebruch polynomial, or sometimes (less descriptively) as the i^{th} L. For each i, this polynomial can be expressed in the Pontryagin classes of the manifold's tangent bundle. The Novikov conjecture states that the higher signature is an invariant of the oriented homotopy type of M for every such map f and every such class x, in other words, if $h\colon M' \to M$ is an orientation preserving homotopy equivalence, the higher signature associated to $f \circ h$ is equal to that associated to f.

3. The life and works of Michael Freedman

Michael Hartley Freedman (born April 21, 1951) is an American mathematician, at Microsoft Station Q, a research group at the University of California, Santa Barbara. In 1986, he was awarded a Fields Medal for his work on the 4-dimensional generalized Poincaré conjecture. Freedman and Robion Kirby showed that an exotic ^{4} manifold exists.

3.1. **Life and career.** Michael Freedman entered the University of California, Berkeley and after two semesters dropped out[4]. In the same year he wrote a letter to Ralph Fox, a Princeton professor at the time, and was admitted to graduate school so in 1968 he continued his studies at Princeton University where he received Ph.D. degree in 1973 for his doctoral dissertation titled Codimension-Two Surgery, written under the supervision of William Browder. After graduating, Freedman was appointed a lecturer in the Department of Mathematics at the University of California, Berkeley. He held this post from 1973 until 1975, when he became a member of the Institute for Advanced Study (IAS) at Princeton. In 1976 he was appointed assistant professor in the Department of Mathematics at the University of California San Diego. He spent the year 1980/81 at IAS, returning to UC San Diego, where in 1982 he was promoted to professor. He was appointed the Charles Lee Powell chair of mathematics at UC San Diego in 1985. Freedman has received numerous other awards and honors including Sloan and Guggenheim Fellowships, a

MacArthur Fellowship and the National Medal of Science. He is an elected member of the National Academy of Sciences, and a fellow of the American Academy of Arts and Sciences and of the American Mathematical Society. In addition to winning a Fields Medal at the International Congress of Mathematicians (ICM) in 1986 in Berkeley, he was an Invited Speaker at the ICM in 1983 in Warsaw and at the ICM in 1998 in Berlin[5]. He currently works at Microsoft Station Q at the University of California, Santa Barbara, where his team is involved in the development of the topological quantum computer.

3.2. **Generalized Poincaré conjecture.** Here is a summary of the status of the generalized Poincaré conjecture in various settings.

- **Top:** true in all dimensions.
- **PL:** true in dimensions other than 4; unknown in dimension 4, where it is equivalent to Diff.
- **Diff:** alse generally, true in some dimensions including 1,2,3,5, and 6. First known counterexample is in dimension 7. The case of dimension 4 is equivalent to PL and is unsettled (as of 2019).

A fundamental fact of differential topology is that the notion of isomorphism in Top, PL, and Diff is the same in dimension 3 and below; in dimension 4, PL and Diff agree, but Top differs. In dimension above 6 they all differ. In dimensions 5 and 6 every PL manifold admits an infinitely differentiable structure that is so-called Whitehead compatible.

3.3. **History.** The case n = 1 and 2 has long been known, by classification of manifolds in those dimensions.

For a PL or smooth homotopy n-sphere, in 1960 Stephen Smale proved for $n \geq 7$ that it was homeomorphic to the n-sphere and subsequently extended his proof to $n \geq 5$; he received a Fields Medal for his work in 1966. Shortly after Smale's announcement of a proof, John Stallings gave a different proof for dimensions at least 7 that a PL homotopy n-sphere was homeomorphic to the n-sphere using the notion of "engulfing". E. C. Zeeman modified Stalling's construction to work in dimensions 5 and 6. In 1962, Smale proved a PL homotopy n-sphere was PL-isomorphic to the standard PL n-sphere for n at least 5. In 1966, M. H. A. Newman extended PL engulfing to the topological situation and proved that for $n \geq 5$ a topological homotopy n-sphere is homeomorphic to the n-sphere.

Michael Freedman solved the case $n = 4$ (in Top) in 1982 and received a Fields Medal in 1986.

Grigori Perelman solved the case $n = 3$ (where Top, PL, and Diff all coincide) in 2003 in a sequence of three papers. He was offered a Fields Medal in August 2006 and the Millennium Prize from the Clay Mathematics Institute in March 2010, but declined both.

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HISTORY OF THE POINCARÉ CONJECTURE

JORDAN O'KEEFFE, ANDREJ KOZLOV, AND JEROEN O'FLAHERTY

Abstract. This paper presents a history of the Poincaré conjecture from it's conception in 1904 to it's first proof presented by Grigori Perelman in 2006.

1. Introduction

On November 13 2002 Grigori Perelman posted the first of three eprints online containing his proof of the Poincaré conjecture. This proof marked the beginning of the end of a long 104 year long journey involving thousands of Mathematicians, from Henri Poincaré's conception of the conjecture in 1904 until the acceptance of Perelman's proof in 2006. The relatively unknown Perelman was cast into the limelight and has since became known as one of the great mathematicians of the modern age.

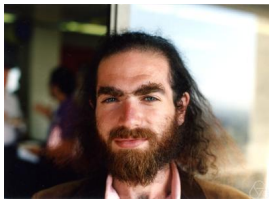

Figure 1. A photo of Grigori Perelman

2. Henri Poincaré's Conjecture

Henri Poincaré was one of the first mathematicians to explore the area of mathematics now known as Topology. Poincaré was investigating the solutions to the algebraic equation $f(x, y, z) = 0$ where $x, y, z \in \mathbb{C}$. Poincaré utilized the notion of assigning algebraic invariants to geometric objects as a way of classifying these geometric objects. This notion is often considered the beginning of algebraic topology.

During his analysis, in 1904 Poincaré conjectured that every simply connected closed three-manifold is homeomorphic to the three-sphere $\mathbb{S}^3$. In the vernacular the conjecture states that the three-sphere is the only type of bounded three-dimensional space possible that contains no holes.

FIGURE 2. A photo of Henri Poincaré

3. Attempted Solutions

The Poincaré conjecture seemed to be forgotten by mathematicians until British mathematician J. H. C. Whitehead began to investigate it during the 1930's and claimed the first proof but later retracted it. In the process of his investigation he discovered examples of simply-connected non-compact 3-manifolds not homeomorphic to $\mathbb{R}^3$ which is now called the Whitehead manifold.

In the decades that followed many influential mathematicians such as Georges de Rham, R. H. Bing, Wolfgang Haken, Edwin E. Moise, and Christos Papakyriakopoulos attempted to prove the conjecture. All of the attempted proofs contained subtle flaws, however in 1954 R. H. Bing proved a weak version of the Poincaré conjecture: if every simple closed curve of a compact 3-manifold is contained in a 3-ball, then the manifold is homeomorphic to the 3-sphere. He published his proof in the Princeton journal Annals of Mathematics. In 1978 in the Mathematical Communications journal Polish mathematician Włodzimierz Jakobsche showed that if the Bing–Borsuk conjecture is true in dimension 3, then the Poincaré conjecture must also be true. In 2007, Israeli–Swiss author George Geza Szpiro published a non-techincal book called Poincaré's prize which details attempts at a proof of the Poincaré conjecture.

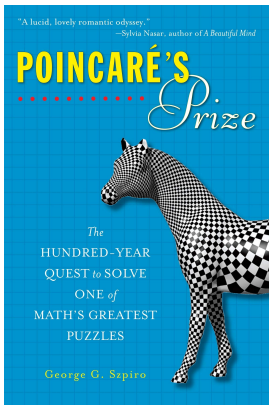

FIGURE 3. Cover of Poincaré's Prize

4. The Millennium Prize Problems

On the 24th of May 2000, the Clay Mathematics Institute announced that they would give a $1 million prize if a correct solution to one of seven problems in mathematics were to be found. The seven problems are:

- The Poincaré conjecture
- P versus NP
- Hodge conjecture
- Riemann hypothesis
- Yang–Mills existence and mass gap
- Navier–Stokes existence and smoothness
- Birch and Swinnerton-Dyer Conjecture

As of March 2020 all of the problems, with the exception of the Poincaré conjecture, remain unsolved.

5. Ricci Flow

Grigori Perelman's proof of the Poincare conjecture used the notion of Ricci Flows. In this section we will discuss what the Ricci flow is.

5.1. **What is the Ricci flow?** A Ricci Flow is an intrinsic geometric flow which deforms the metric of a Riemannian mannifold in a way which is analogous to the diffusion of heat. Mathematically the normalized Ricci flow is given by:

$$\partial_t g_{ij} = -2R_{ij} + \tfrac{2}{n} R_{avg} g_{ij}$$

Where g_{ij} is a the metric tensor for the Riemannian manifold, R_{ij} is the Ricci tensor and R_{avg} is the mean of the scalar curvuture and n is the dimension of the manifold.

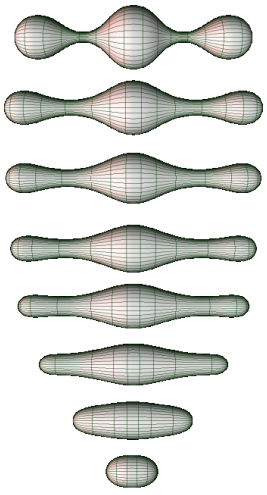

Figure 4. Evolution of the Ricci Flow

6. Grigori Perelman's Proof

On November 13th 2002 Perelman uploaded the first of three eprints containing his proof of the Poincaré conjecture. However, it was not until 2006 that his proof was reviewed and confirmed as correct. Upon the announcement that his proof was correct Perelman was awarded the $1 million and the prestigious Fields Medal. His proof was also awarded as the breakthrough of the year from the journal Science, the first time it was awarded to a mathematical breakthrough. In an unexpected act, Perelman declined both the $1 million prize and the Fields medal. When questioned on his act Perelman delcared:

"I'm not interested in money or fame; I don't want to be on display like an animal in a zoo."

Figure 5. A more recent photo of Perelman

Figure 6. The prestigious Fields Medal which Perelman declined

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KNOT THEORY

PADRAIG O'SHEA, EDWARD MCANDREW, AND EVAN O'BRIEN

Abstract. Throughout our study of topology we have came across the theory of knots. In this project we will briefly be discussing some fundamentals of Knot Theory. We have split knot theory into three subsections;what is a knot?, the main objectives of knot theory and the origins of knot theory.

1. What is a Knot?

Knot theory in mathematics is the study of closed curves in three dimensions, and their possible deformations without one part cutting through another. Knots are formed by interlacing and looping a piece of string in any fashion and then joining the ends just like tying a shoelace. There are two questions that arise when we study knots. The first question is whether such a curve is truly knotted or can simply be untangled? In other words, whether or not one can deform it in space into a standard unknotted curve like a circle. The second question is whether any two given curves represent different knots or are they the same knot in the sense that one can be continuously deformed into the other?

2. The Fundamentals of Knot Theory

At its most basic, knot theory considers lines embedded in 3-dimensional space. The most basic result in knot theory is proving that the Trefoil knot cannot be deformed continuously to a circle without cutting.

This is a very nice problem because it can be easily understood and is an example to start off with when teaching an introductory knot theory class. Also, it is not equivalent to its mirror image as was proved by Max Dehn in 1914. If two knots are equivalent then this can be proved by a sequence of Reidemeister moves, but the known upper bounds on the number of steps are extraordinarily large. Yet this proves that testing equivalence is decidable, something non-trivial. If we want to prove that two knots K1 and K2 are not equivalent then a basic method would be to use an invariant(inv) with inv(K1) not being equal to inv(K2). One of the main

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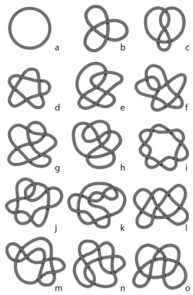
goals of knot theory has always been to identify properties that truly distinguish knots—to find what are known as knot invariants. A knot invariant acts very much like a "fingerprint" of the knot; it does not change by superficial deformations of the knot. At first glance, you may think that the minimum number of crossings in a knot could serve as such an invariant. After all, no matter how hard you try, you will never be able to reduce the number of crossings of the trefoil knot to fewer than three. However, the minimum number of crossings is actually not a very useful invariant. As the photo below illustrates, there are three different knots with six crossings and no fewer than seven different knots with seven crossings.



3. Classification and Complexity of Knots

The basic tool for classifying knots consists of projecting each knot onto a plane picture. We then count the number of times the projected shadow of the knot crosses over itself noting at each crossing which direction goes "over" and which goes "under." A measure of the knot's complexity is the least number of crossings that occur as the knot is moved around in all possible ways. The simplest possible true knot is the trefoil knot, or overhand knot, which has three such crossings. The order of this knot is therefore 3. Even this simple knot has two configurations that cannot be deformed into each other, although they are mirror images. There are no knots with fewer crossings, and all others have at least four. The number of distinguishable knots increases rapidly as the order increases. For example, there are almost 10,000 distinct knots with 13 crossings, and over a million with 16 crossings, which is the highest known by the end of the 20th century. Certain higher-order knots can be resolved into combinations, called products, of lower-order knots. For example, the square knot and the granny knot (sixth-order knots) are products of two trefoils that are of the same or opposite chirality, or handedness. Knots that cannot be so resolved are called prime. A knot that is the equivalent of its mirror image is called amphicheiral.

4. Definition of the Equivalence of Knots

Definition of the equivalence of knots (mathematic form): Let h :Kx[0, 1] tends to R3 a homotopy. If t is an element of [0, 1] indicates time, then ht(K) for increasing values of t show the evolution of K in R3 (h0 is the identity map, all ht are continuous). Homotopy allows the knot to intersect itself, so it is not useful! Why? All knots are homotopic to the trivial knot! Let all ht injective (i.e. h is an isotopy). Isotopy allows to shrink a part of the knot down to a point, so it is not useful because all knots are isotopic to the trivial knot. We need to consider h an

ambient isotopy, which allows to deform the knot through the space R 3 in which it sits in i.e. an ambient isotopy of K, a set R3 is an isotopy that carries K with it.

5. Examples of Knots

Unknot (above)

Torus Knot

Figure of Eight Knot

6. The Origins of knot theory

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The first traceable mathematical theory relating to knot theory was in c.1883 by a German mathematician called Carl Friedrich Gauss. Who developed the Gauss linking integral for computing the linking number of two knots. However, the true origins modern day knot theory can be accredited to a Scottish mathematical physicist William Thompson, commonly referred to as Lord Kelvin through a theory proposed in 1869. This theory speculated that atoms potentially consist of knotted vertex tubes of one another, with different elements corresponding to different knots. From this theory, another Scottish mathematical- physicist called Peter Guthrie Tait made strides by making the first symmetric attempt to classify knots. While Lord Kelvin's theory was eventually rejected it started further research into knot theory, that developed into a pure mathematical theory. In the early 20th century the development of topology led by Henri Poincaré, caused mathematicians such as J.W. Alexander, Kurt Reidemeister and Max Dehn to delve further into the study of knots. From these investigations grew theories such as Reidemeister moves, the Alexander polynomial and also Dehn surgery which drew a relation between knots and the general theory of 3-manifolds. A notable pioneer in knot theory in the first half of the 20th century was Ralph Fox who popularized knot theory. A number of major discoveries in the late 20th Century helped rejuvenate interest into knot theory and bring it back into the forefront of mathematics. In the late 1970's a mathematician called William Thurston released the theory of hyperbolic 3-manifolds, which proved to be of vital importance. Due to his developments William Thurston was awarded a Fields Medal for his work in 1982. In 1984 Vaughan Jones' discovered the Jones polynomial, which was instrumental in

finding other knot polynomials, for example; the bracket, Homfly and Kauffman polynomials. As well as Thurston, Vaughan Jones' was awarded the Fields Medal in 1990. In 1988 Edward Witten proposed an alternate framework for the Jones polynomial, by adapting existing ideas and introducing new concepts such as topological quantum field theory. For this work Witten also received the Field's Medal in 1990. In the last several decades of the 20th Century, following the resurgence in knot theory, scientists and mathematicians began to find other useful applications of knot theory in other fields such as biology and chemistry, proving knot theory to be instrumental in the scientific world we currently live in.

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Knot Theory and its Origins

Matthew Keating
Liam Rooney

March 2020

1 Introduction

Knot theory was first theorised as an explanation for what atoms were made up of in the ether. We know now that this is untrue, however in their attempt to study these knots they formulated some important questions for knot theory. What makes one knot different from another? How can we prove two knots are different?

A knot is an embedding of a circle into a 3D space, however many times we see knots represented in 2D diagrams to allow them to be easily represented on paper. The simplest knot is one with no crossings and is known as the 'Unknot'.

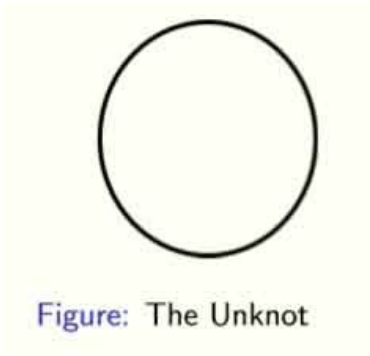
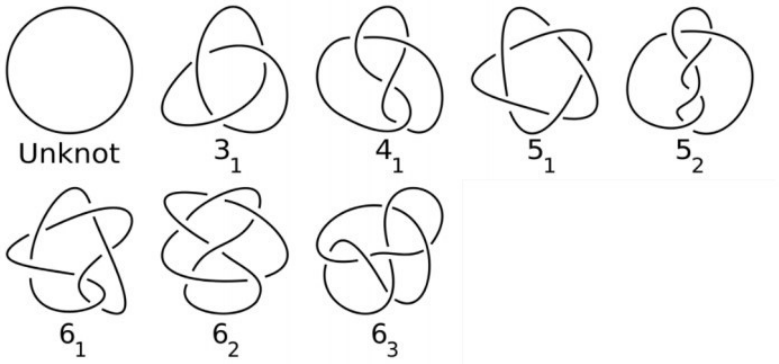


Figure: The Unknot

To distinguish between different knots we give them a number notation with the form A_B. "A" represents the crossing number, i.e. the minimum number of crossings given in any knot diagram. "B" represents the Index Number, an arbitrary index assigned to a knot to distinguish it from a different knot with the same crossing number. Here are some examples:



Knot 3_1 is also commonly known as the 'trefoil' knot, and 4_1 is sometimes referred to as the 'figure 8' knot. You can see in the case of 6_1, 6_2 and 6_3, all three knots contain the same number of crossings, however these knots are not the same, this is why the index number is necessary.

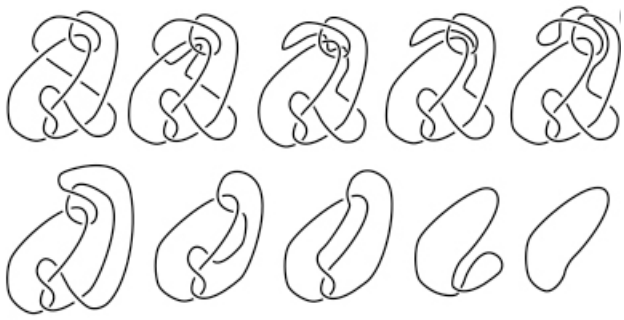
This brings us to our next question; how do you know when two knots are the same?

2 Knot Equivalence and The Three Reidemeister Moves

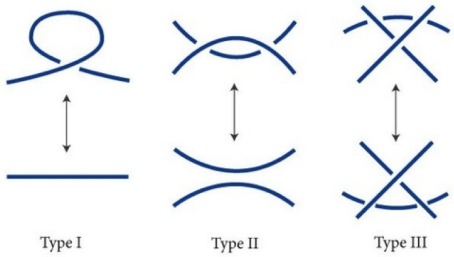
We consider two knots to be the same if you can deform one into the other without passing through itself. If two knots are the same, they are referred to being ambient isotopy.

Let's look at our most basic knot once again the Unknot and a trickier looking knot 'The Culprit Knot'.

Although these knots look completely different, they are both in fact the Unknot. The Culprit Knot can be transformed back into the Unknot through transformations known as Reidemeister moves. It takes ten of these Reidemeister moves.



In the 1930s, Kurt Reidemeister proved that all deformations can be reduced into 3 moves: the 'Twist' (RI), the 'Poke' (RII) and the 'Slide' (RIII).



The Reidemeister moves are a great way of proving two knots are the same, however how can it be used to prove two knots are different? As one person could spend hours trying to twist poke and slide a knot (for example, the culprit knot) to try transform it into a different type of knot (unknot) and never figure out how to do it, even though it is possible. We need to find a more convenient way to prove two knots are not the same. For this we will introduce an extremely helpful tool known as a knot invariant.

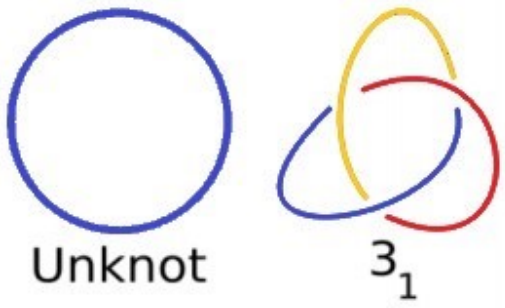
3 Knot Invariants

Knot invariants are unchanging characteristics of a knot, in other words they are unaffected by the Reidemeister moves. A simple example invariant that can be used is known as tricolourability. Tricolourability is a knot's ability to be coloured with three different colours, subject to certain rules. This will in turn allow us to easily distinguish between two different categories of knots, tricolourable and non-tricolourable.

The rules for a knot to be tricolourable are as follows;

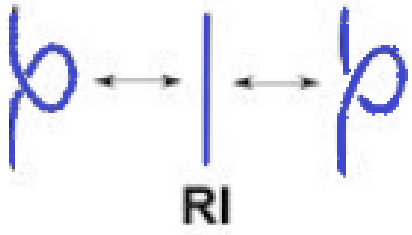
- 1. At least two colours must be used (no more than 3).
- 2. At each crossing, the three incident strands must all be the same colour, or three different colours.

Now let's see if our most basic knots, the Unknot and the Trefoil, are tricolourable.



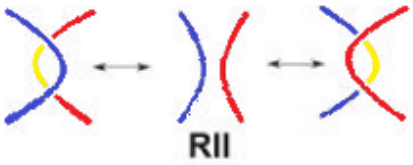
As we see the Trefoil (3_1) passes all our criteria, it has at least 2 colours and at every crossing all three colours are used. However, we see that the unknot does not meet the first criteria, it only uses one colour. Therefore, we now know that the Trefoil is tricolourable and the Unknot is non-tricolourable. To prove that these two knots are different we must prove that our invariant is unaffected by the Reidemeister moves.

Move 1: Twist



We see the effect of the twist has not broken any of our criteria, as at the crossing all the strands are the same colour.

Move 2: Poke



Again, we see none of our criteria have been broken by the second Reidermeister move as all crossings have all three colours.

Move 3: Slide

All crossings use all three colours and therefore our criteria have been met.

Hence, the Reidermeister moves cannot affect a knots tricolourability.

So, given this we are now sure any knot that is non-tricolourable is different then any knot that is tricolourable.

As this is such a simple invariant it only divides our knots into only two categories. Two knots being both tricolourable or non-tricolourable does not mean they are the same, therefore we would now have to use more knot invariants to further distinguish between these knots. Our example is just to show how a basic knot invariant is used to prove two knots are different.

4 Origins of Knot Theory

Knot theory first came about as a physicist's erroneous concept as a model for the atom. Although this model has since been rejected, the study of knot theory has developed from this model into a promising field of mathematical exploration in its own right.[1]

There had been previous mentions of the ability to view knots as mathematical entities, but not until the 19th century were the first inroads made into what we now know as Knot Theory by Carl Friedrich Gauss. Among Gauss' notes was a collection of drawings dated 1974. Gauss drew some knots, labelling the crossings with letters and then noted the sequence of letters one would encounter as they travelled from an arbitrary starting point along the knot until they arrived back to the start. If the knot had n crossings, the sequence would be of length 2n. Through his later study of electrodynamics, Gauss derived what is now referred to as the "Gauss Linking Number". The linking number is the numerical invariant that describes the linking of two closed curves in three-dimensional space. It represents the number of times that each curve winds around the other. The number does not change under smooth deformation of the two loops. Therefore, it is unchanging under ambient isotopy. This was the first method developed to distinguish two non-equivalent links from each other.[1]

A physicist by the name of William Thompson (Lord Kelvin) made further discoveries regarding Knot Theory during the 19th Century. At the time, scientists were divided by two factions of thought; those who believed "corpuscular theory" (matter is composed of atoms) and those who supported the theory that matter consisted of waves. Thompson was attempting to combine

these two ideas into one theory. Thompson was following on from Hermann von Helmholtz' idea that that there existed an all permeating medium he called the "ether" [2]. Helmholtz concluded that vortices of ether (which he believed to be an ideal fluid), were stable. These vortices could become knotted and still retain their original identities. From this, Thompson theorized that matter was composed of three-dimensional knotted tubes of ether, or "vortex atoms". He proposed that different elements were the result of different twisting and crossing formations within these knotted tubes.[1]

Thompson's good friend, James Clerk Maxwell, followed on from these studies, interested in the idea that knots could be used in the study of electricity and magnetism. In Maxwell's diagrams of knots, he distinguished over and under crossings. He then pondered how one could recreate his diagrams, without changing the properties of the knots. Through this he discovered what would later be known as the three Reidemeister moves.

Another friend of Thompson, Peter Guthrie Tait, began creating the first table of knots in 1867, as along with Thompson's vortex atoms theory came the need for a system of classifying knots. A mathematician named Thomas Kirkman continued the research of knot classification. Using an operation similar to Reidemeister's second move, he simplified his diagrams in order to minimise the number of duplicates. Tait then made further progress in this field, working with Charles Newton Little. Together they developed a method for the notation of knots which eliminated ambiguity, differentiating between knots which had the same number of crosses yet were clearly distinct. This was a difficult task as although they could spot knots which were unique from each other, there was no way to tell if knots were equivalent except by visual examination. A passage from one of Tait's journals reads "... though I have grouped together many widely different but equivalent forms, I cannot be absolutely certain that all those groups are essentially different from one another". The pair produced the first official table of alternating knots with up to ten crossings.[1]

This early work paved the way for later mathematicians to develop the knot theory we now know. Although some of this work has since been disproved, without it we would not have as much knowledge on the topic today. Knot tables have now been extended to include all prime knots with 16 or fewer crossings, over 1.7 million unique knots.[3]

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TOPOLOGY PAPER ON FIELDS MEDAL

KATIE SEXTON AND CAOIMHE FLEMING

MA342 Topology Lecturer: Graham Ellis

Abstract. This paper outlines the details of the fields medal and discusses the work of some of the remarkable mathematicians who have won the prestigious award

What is The fields medal?

The fields medal, often regarded by mathematicians as the most prestigious award one can receive, is awarded at the International Mathematical Union (IMU) to two three or four mathematicians who must be under the age of forty. The IMU is a meeting that takes place every four years. The award is sometimes considered to be like the Nobel Prize of the mathematical award however there are a few key differences including age differences, how regularly they can be awarded as well as number of awards. A mathematician who receives the award also receives a monetary prize of 15000 Canadian Dollars. The name of the award is in honour of John Charles Fields; a mathematician who was instrumental in the establishment of the award as well as sponsoring the monetary component and even helping to design the medal itself. Fields began planning the award in the late 1920s but, due to deteriorating health, never saw the implementation of the medal in his lifetime. He died on August 9, 1932 after a three-month illness; in his will, he left $47,000 for the Fields Medal fund.$

The Life and Career of John Charles Fields

Born in Hamilton, Ontario, to a leather shop owner, Fields graduated from Hamilton Collegiate Institute in 1880 and the University of Toronto in 1884 before leaving for the United States to study at Johns Hopkins University in Baltimore, Maryland. Fields received his Ph.D. in 1887. His thesis, entitled Symbolic Finite Solutions and Solutions by Definite Integrals of the Equation dny/dxn = xmy, was published in the American Journal of Mathematics in 1886. Fields taught for two years at Johns Hopkins before joining the faculty of Allegheny College in Meadville, Pennsylvania. Disillusioned with the state of mathematical research in North America at the time, he left for Europe in 1891, locating primarily in Berlin, Göttingen and Paris, where he associated with some of the greatest mathematical minds of the time, including Karl Weierstrass, Felix Klein, Ferdinand Georg Frobenius and

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Max Planck. Fields also began a friendship with Gösta Mittag-Leffler, which would endure their lifetimes. He began publishing papers on a new topic, algebraic functions, which would prove to be the most fruitful research field of his career. Fields returned to Canada in 1902 to lecture at the University of Toronto. Back in the country of his birth, he worked tirelessly to raise the status of mathematics within academic and public circles. He successfully lobbied the Ontario Legislature for an annual research grant of 75,000 dolars for the university and helped establish the National Research Council of Canada, and the Ontario Research Foundation. Fields served as president of the Royal Canadian Institute from 1919 until 1925, during which time he aspired to mold the institute into a leading centre of scientific research, although with mixed success. His efforts, however, were pivotal in making Toronto the location of the 1924 International Congress of Mathematicians (ICM). He was an Invited Speaker of the ICM in 1912 at Cambridge, in 1924 at Toronto, and in 1928 at Bologna.

Controversies and how the recipients are chosen

The most famous prize in mathematics, the Fields Medal, is often described as the Nobel Prize for math. Now, confidential correspondence from the 1950s provides a first window into the deliberations of early Fields Medal committees. The letters suggest that the award was never intended to honour the most important discoveries in the field but was meant to recognise promising up and coming talents. "It's basically exactly the opposite of how the Fields works today" In 1966, the organization (now run by the International Mathematics Union) bumped it up to four medals every 4 years and stipulated that recipients must be younger than 40. Yet the award committee's criteria for selecting the winners are notoriously enigmatic. Nowadays, deliberations are sealed for a period of 75 years, whereas records from earlier years weren't kept in any organized way. Everyone agreed the medals should go to relatively young mathematicians, but debate raged over the definition of "young." The committee also disagreed on whether the prizes should simply go to the most talented young mathematicians or to those who were comparatively unrecognized. Committee chair Heinz hopf at the Swiss Federal Institute of Technology in Zurich ruled out two nominees because they were already thriving professionally. The U.S. mathematician John Nash, whose life was dramatized in the movie "A Beautiful Mind", came in third place in votes, Years later Nash openly wondered why he hadn't won for his brilliant contributions to solving partia differential equations In the minds of the committee members, Barany says, Nash didn't need the Fields medal to be successful, and they preferred recognising the comparatively lesser known Roth and Thom. "I was fascinated by the way in which personal agendas of committee members were clothed in such seemingly reasonable attempts to place restrictions on the prize" says Christopher Hollings, a historian of mathematics at the University of Oxford in the United Kingdom, who attended Barany's talk "it is a nice and interesting reminder that mathematicians are people too"

LANDMARK WINNERS

The fields medal was first awarded in 1936 to the American mathematician Jesse Douglas as well as Lars Ahlfors from Finland. Since then it has been awarded every four years with the aim of recognising and supporting young mathematicians who

have made important contributions to the subject. In 1954 at the age of 27 Jean-Pierre Serre became the youngest person to win the award for his work on algebraic topology and to date remains the youngest field medallist. Maryam Mirzakhani was the first woman to receive the award in 2014.

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1. Youngest Winner: Jean-Pierre Serre

From a very young age he was an outstanding figure in the school of Henri Cartan, working on algebraic topology, several complex variables and then commutative algebra and algebraic geometry, where he introduced sheaf theory and homological algebra techniques. Serre's thesis concerned the Leray–Serre spectral sequence associated to a fibration. Together with Cartan, Serre established the technique of using Eilenberg–MacLane spaces for computing homotopy groups of spheres, which at that time was one of the major problems in topology. In the 1950s and 1960s, a fruitful collaboration between Serre and the two-years-younger Alexander Grothendieck led to important foundational work, much of it motivated by the Weil conjectures. Two major foundational papers by Serre were Faisceaux Algébriques Cohérents (FAC),[3] on coherent cohomology, and Géometrie Algébrique et Géométrie Analytique (GAGA).[4] Even at an early stage in his work Serre had perceived a need to construct more general and refined cohomology theories to tackle the Weil conjectures. The problem was that the cohomology of a coherent sheaf over a finite field couldn't capture as much topology as singular cohomology with integer coefficients. Amongst Serre's early candidate theories of 1954–55 was one based on Witt vector coefficients. Around 1958 Serre suggested that isotrivial principal bundles on algebraic varieties – those that become trivial after pullback by a finite étale map – are important. This acted as one important source of inspiration for Grothendieck to develop the étale topology and the corresponding theory of étale cohomology.[5] These tools, developed in full by Grothendieck and collaborators in Séminaire de géométrie algébrique (SGA) 4 and SGA 5, provided the tools for the eventual proof of the Weil conjectures by Pierre Deligne.

From 1959 onward Serre's interests turned towards group theory, number theory, in particular Galois representations and modular forms. Amongst his most original contributions were: his "Conjecture II" (still open) on Galois cohomology; his use of group actions on trees (with Hyman Bass); the Borel–Serre compactification; results on the number of points of curves over finite fields; Galois representations in -adic cohomology and the proof that these representations have often a "large" image; the concept of p-adic modular form; and the Serre conjecture (now a theorem) on mod-p representations that made Fermat's last theorem a connected part of mainstream arithmetic geometry. In his paper FAC, Serre asked whether a finitely generated projective module over a polynomial ring is free. This question led to a great deal of activity in commutative algebra, and was finally answered in the affirmative by Daniel Quillen and Andrei Suslin independently in 1976. This result is now known as the Quillen–Suslin theorem

In his speech at the Fields Medal award ceremony in 1954, Hermann Weyl gave high praise to Serre, and also made the point that the award was for the first time awarded to a non-analyst. Serre subsequently changed his research focus.

1.1. **The Serre spectral sequence.** In mathematics, the Serre spectral sequence (sometimes Leray-Serre spectral sequence to acknowledge earlier work of Jean Leray

in the Leray spectral sequence) is an important tool in algebraic topology. It expresses, in the language of homological algebra, the singular (co)homology of the total space X of a (Serre) fibration in terms of the (co)homology of the base space B and the fiber F. The result is due to Jean-Pierre Serre in his doctoral dissertation.

2. First female winner : Maryam Mirzakhani

Mirzakhani was awarded the Field's Medal for her "her outstanding contributions to the dynamics and geometry of Riemann surfaces and their moduli spaces". Maryam Mirzakhani was reinterested in closed geodesics on a hyperbolic surface. These are closed curves whose length cannot be shortened by deforming them. The prime number theorem for geodesics states that the number of closed geodesics grows exponentially with L; it is asymptotic to $(e^L)/L$ for large L Mirzakhani focused on what happens to the "prime number theorem for geodesics" when one considers only closed geodesics that are simple (they do not intersect themselves). The behaviour of these geodesics' changes, the growth of the number of geodesics of length at most L is no longer exponential in L but is of the orderL^6g-6, where g is the genus of the surface. Mirzakhani showed that the number is asymptotic to $c \cdot L^6g-6$ for L going to infinity, where c depends on the hyperbolic structure. Although this statement was about a single, hyperbolic structure, Mirzakhani proved it by contemplating all such structures simultaneously. The complex structures on a surface of genus g form a continuous, or non-discrete, space, since they have continuous deformations. (While the underlying topological surface remains the same, its geometric shape changes during a deformation. Riemann knew that these deformations depend on $6g6$ parameters or "moduli", meaning that the "moduli space" of Riemann surfaces of genus g has dimension 6g 6. 0Mirzakhani determines a link between the volume calculations on moduli space and the counting problem for simple closed geodesics on a single surface by using work by G. McShane. This allowed her to calculate certain volumes in moduli space and then infer the counting result for simple closed geodesics.

Design of the medal

The design of the medal displays Archimedes as well as a latin quote attributed to him which reads "Transire suum pectus mundoque potiri" (Rise above oneself and grasp the world. The date is written in roman numerals however it contains one error. It should read MCMXXXIII but instead says MCNXXXIII. It also has the words "of Archimedes" written in Greek capital letters. On the reverse is the inscription in Latin CONGREGATI In the background, a representation of Archimedes' tomb can be seen, with the carving illustrating his theorem on the sphere and cylinder behind an olive branch. (This is the mathematical result of which Archimedes was reportedly most proud: Given a sphere and a circumscribed cylinder of the same height and diameter, the ratio between their volumes is equal to .) The name of the prize-winner is inscribed on the rim of the medal.

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KNOT THEORY

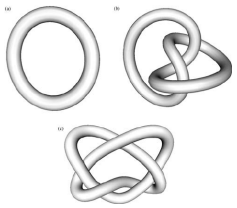
CIAN TIGHE, LUKE WARD AND SARAH GIBBONS

Abstract. This essay will give an introduction to Knot Theory as well as its definition. Examples of how the theory can be used will be shown and some properties of the theory will be discussed.

1. Introduction

Knot theory is an engaging topic as we are familiar to the objects studied within it in our everyday live, from different structures to DNA. In topology, knot theory is the study of mathematical knots which differ from knots in real life in that the knot can not be undone because its ends are joined together.

2. What is a Knot?



The above diagrams are all knots in their own way. But how do we define what is a knot, what is not a knot (in a mathematical sense) and what is the different between these "mathematical knots" and knots we know from day-to-day life? The first knots that probably spring to mind are the knots we use when tying our shoelaces. However, this is not a "mathematical knot" because it has loose ends, having loose ends means it can be untied to form a different knot, it is not topologically "locked in". In mathematics, a knot is the embedding of a 2d circle in R^3, the difference between this and a conventional knot is that a mathematical knot is "closed" ie. there are no ends to tie or untie.

3. Types of Knots

We classify knots by its "crossing number". **A Crossing Number** of a knot K, denoted $c(K)$, is the least number of crossing that occur, ranging over all possible diagrams.

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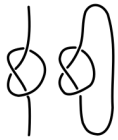
Crossing Number	Number of Prime Knots
3	1
4	1
5	2
6	3
7	7
8	21
9	49
10	165

Type of "Well-Known" Knots

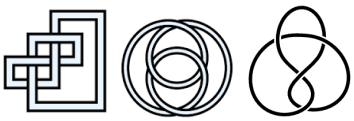
- **Ring Knot - "Unknot"** - The most simplest knot known to us is the ring knot, also known as the "unknot" (Knot theory, 2020) which is homeomorphic to the circle and a subspace of euclidean three-dimensional space (Armstrong, 2010). We also call this the "trivial knot" due to its simplistic nature. One way of visualizing this is an elastic band on a flat surface. No matter how we manipulate the band - we will always be able to return to the trivial knot. This is known as Knot 0_1 (0 for the minimal number of crossings, the 1 a serial index telling us it is a first kind of knot with 0 crossings). For different variants (see how the unknot turns to a heart-like shape), we can visualize how this is equivalent to the "unknot" by creating the same shape with a single rubber band and returning it back to its original form, we wouldn't be able to do the same with a different knot.

- **Trefoil Knot** - The trefoil knot is perhaps the simplest example of a non-trivial knot. From the name, we can deduce easily that it likely only has three crossings - and we would be right. In popular culture, a lot of people would recognize it as Adidas' original logo. We obtain a trefoil knot by taking a overhand knot and connecting the loose ends together.



We notate the trefoil knot as 3_1 (the trefoil knot is the only knot with crossing number 3). Since there are no non-trivial knots with 1 or 2 crossings, the trefoil is the first non-trivial knot. The trefoil appears along in religious symbols - due to Christianity's link with the number three (to denote the "Holy Trinity"). In popular culture, a lot of people would recognize it as Adidas' original logo. We obtain a trefoil knot by taking a overhand knot and connecting the loose ends together.

- **Figure-8 Knot** - is the only knot with four crossings. It gets its name from the distinct 8 we can see in the knot. It is notated has 4_1 and can come in many different forms but all are topologically the same.

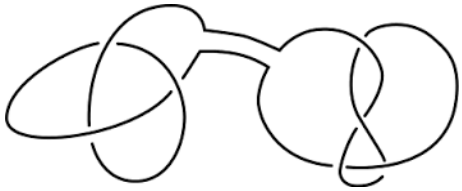


4. Mathematical Definitions

Mathematically, a simple knot can be defined as a simple closed polygonal curve in R^3 and represented by the ordered set (p_1,p_2,...,p_n) joined by the line segments [p_1,p_2],[p_2, p_3],...,[p_{n1}], [p_n, p_1] (De Santi, 2002).

If an ordered set defines a knot and there is no subset which defines the same knot as the order set then the elements of the order set p_1, p_2... are named the **vertices of the knot**.

The knot can then be represented in a knot diagram with the use of these vertices and projections. **A regular projection** occurs if three points on the knot do not project on to the same point as well as no vertex projecting to the same point as any other point. The following theorem can be used when dealing with knot projections and equivalency: *If a knot does not have a regular projection then there is an equivalent knot that does have a regular projection.* Two knots, say k_1 and k_2, are said to be equivalent if there exists a homeomorphism of $\mathrm{E}^3 s.t. h(\mathrm{k}_1)=k_2$ (Armstrong, 2010).



The connected sum of two knots K_1 and K_2, is formed by removing a small arc form each knot and then connecting the four endpoints by the two new arcs in such a way that no new crossings are introduced, resulting in a single knot. K = K_1 + K_2. Above is the connected sum of a figure-8 knot and a trefoil knot.

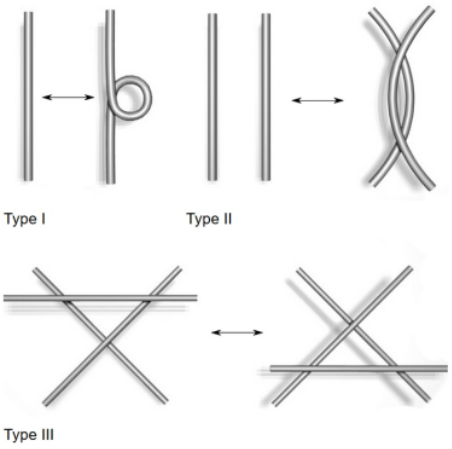
Another important definition in knot theory is the **orientation of knots**. Knowing the orientations of knots allows us define the sum of oriented knots and are essential for using the Seifert algorithm to produce Seifert surfaces (MathWorld, 2020). Reversing the orientation of a knot K can possibly give rise to an in-equivalent knot (MathWorld, 2020). **An oriented knot** is defines as a knot and an ordering of its vertices in which the ordering must be chosen so that it imposes the original knot.

If two orderings differ by a cyclic permutation they are said to be equivalent to one another (De Santi, 2002).

5. Equivalence - What makes Knots "the same"?

Two knots are equivalent if it can be jiggled around, tangled or untangled until they are the same. It is difficult to know by looking at the Knots whether they are equivalent or not. However, in Knot Theory the development of Reidemeister moves aids us in this decision.
What is a Reidemeister move? - A Reidemeister move is an operation that can be performed on the diagram of a knot without altering the corresponding knot (De Santi, 2002), it was derived in 1927 by mathematician Kurt Reidemeister. There are three types of Reidemeister moves.

Type I: Twist and Untwist in either direction.
Type 2: Move one loop over another.
Type 3: Move a string completely over or under a crossing.



Theorem - If two knots are equivalent, their diagrams are related by a sequence of Reidemeister moves.

6. Colourable Knots

A knot diagram is called colourable if each arc can be drawn using one of three colours in such a way that:

- At least two of the colours are used.
- At each crossing either three different colours come together or all the same colour comes together.

However we can colour the figure-8 is such a way - therefore it is not colourable. Also, clearly, the unknot is not colourable because its standard projection cannot be coloured. ∴ any colourable knot is non-trivial.

If a diagram of a Knot K is colourable, then every diagram of K is colourable.
To gain a greater understanding of colourability, we take a look at modp labeling.

7. MODP LABELLING

A knot diagram can be labeled modp if each edge can be labeled with an integer from 0 to $p-1$ such that:

- At least two labels are distinct.
- At each crossing the relation $2x - y - z = 0$ (mod p) holds, where x is the label on the overcrossing and y and z the other two labels.

If some diagram for a knot K can be labeled modp then every diagram of K can be labeled modp.
Like colourability, we can gain a better understanding of mod p labelling by the introduction of another topic.

8. GROUP LABELLING

A labelling of an oriented knot diagram with elements of a group consists of assigning an element of the group to each arc of the diagram, subject to the following two conditions:

- At each crossing of the diagram three arcs appear, each of which should be labeled with an element from the group. The label of the arc that passes under the crossing must be conjugate to the one that emerges from the crossing via the label on the over-crossing.
- The labels must generate a group.

If a diagram for a knot can be labeled with elements from a group G, then any diagram of the knot can be so labeled with elements from that group, regardless of the choice of orientation.

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ON CONVERGENCE OF SETS AND DIRECTED SETS IN A TOPOLOGY*

Ronan Timon, Adam Flannery, Paul Brennan
Communicated by Julie Bergner

Abstract. In this article we explore the most common definition for convergence seen initially in any undergraduate mathematics course. We show some simple propositions that may have worked in any analysis done prior to MA342 and show that they are less useful in a topological space. We then offer a better definition after introducing the notion of a net, and show that this makes our propositions rigid again.

1 Introduction

The notion of convergence is one that has appeared many times in undergraduate maths and is often intuitively understood. To consider the notion in Topology however requires some abstractions that are challenging to wrap one's head around, but can be a useful tool when considering some strange functions (such as peano curves). Here we will look at an idea of convergence some here will have already seen, and demonstrate its failing in some more cumbersome topological spaces. We will then introduce the notion of a net and show its validity. We will be dealing a lot with a topology not seen in the MA342 course, the co-countable topology.

Let$X = x_1, ..., x_n$be a set in a topological space.

Definition 1 If X is uncountable then a subset A is called co-countable when its complement is countable.

Definition 2 The collection: $T_{cc} = G \subset X | G$ is co-countable or $G = \phi$ form a topology on an uncountable set X.

It is rather simple to demonstrate this, we will not do this here for brevity' s sake. First we will formalise what most people intuitively understand as convergence.

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Definition 3 A sequence $(X_n), n \in N$ of points in a topological space $(X, \mathcal{T})$ is said to converge to a limit l in X if, For each neighbourhood U of l, there exists $n(U) \in N$ such that $n \leq n(U)$. Then $x_n \in U$

This will look familiar to anyone who has studied metric spaces, and in a metric space this definition is all you need. We will look now at three propositions which can be shown to hold, but their converses fail. We will use (R, T_{cc}) to show this.

2 Propositions and their Converses

Proposition 1 If G is open in $(X, \mathcal{T})$ then, whenever $x \in G$ and x_n goes to x, then $x_n \in [0, 1]$ for large n.
Converse: $[0, 1]$ is not open in T_{cc} , and yet if x_n goes to $x \in [0, 1]$, then $x_n \in [0, 1]$ for large n. The converse does not hold.

Proposition 2 If, in $(X, \mathcal{T})$ there is a sequence of elements belonging to $A \subset X$ and converging to $l \subseteq X$, then $l \in \bar{A}$

Lemma 1 $[0.\bar{1}] = R$ in (R, T_{cc})

Converse: $2 \in [0.\bar{1}]$ and yet no sequence in $[0, 1]$ can have 2 as a limit. The converse does not hold.

Proposition 3 If the map $f : (X, \mathcal{T}) \mapsto (Y, \mathcal{T})$ is continuous, then wherever x_n goes to l in X, then $f(x_n)$ goes to $f(l)$ in Y.

Lemma 2 x_n goes to $x$in(R, T_{cc}), then x_n is eventually constant at x.

Proof: Let $X_n, n \in N$ be a convergent sequence, wherex_n goes to $l \in R$.

$$V := R\{x_n : x_n \neq).V \in T_{cc}$$

Since x_n goes to l, there exists $n(U) \in N$goes to l for $n \geq n(U)$ $x_n \neq V$, the only $x_n = V$ is l. Therefore $x_n = l$, therefore the series is eventually constant.

Converse: The identity map $id : (R, T_{cc})$ goes to (R, T_{usual}), x goes to x is not continuous as $id^{-1}(0, 1) = (0, 1) \neq T_{cc}$. But x_n goes to x in $(R, \mathcal{T}_{cc})$, eventually becoming constant. constant. Therefore $id(x_n)$ goes to $id(x)$ in (E). The converse does not hold.

3 Alteration of the Set

To create something more robust we need only make a modification to he set by which we order our entries. (Which has been N so far).

Definition 4 A Directed Set $(D. \Leftarrow)$ is a non-empty set D, together with a quasiorder (or preorder) $\Leftarrow$ in which two elements possess a common upper bound; i.e. $\Leftarrow$ is reflexive ($x \Leftarrow x$ for all $x \in \mathcal{T}$), transitive ($x \Leftarrow y$, y, then $x \Leftarrow z$) and for every $ab \in D$ there exists $u \in D$ such that $a \Leftarrow u$ and $b \Leftarrow u$.

Some examples. $(N, \Leftarrow)$ and $(N, division)$ The set of all neighbourhoods about $x_0 \in (X, \mathcal{T})$ with reverse inclusion $\Leftarrow$ goes to T

Definition 5 A net in a topological space X is a map from some directed set D into X. We will denote a net as $(x_g), g \in D$, $x_g \in X$ to parallel the traditional notion of sequences.

Definition 6 A net $x_g, g \in D$ of points in a topological space $(X, \mathcal{T})$ is said to converge to a limit $l \Leftarrow XP$, for each neighbourhood U of l there exists $g(U) \in D$ such that $g \geq g(U)$, therefore $x_g \in U$ We still write x_g goes to l.

We can now explore our more robust propositions. It is well within the scope of MA342 to prove both the statements and their converses. But for consistency and to be concise, we will only prove the converse.

Proposition 4 G $\subseteq$(X, T)$isopen iff whenever$x $\in G$ and a net x_g goes to $x, x_g \in G$ for large g.

Lemma 3 Given a point x in a topological space $(X, \mathcal{T})$ and using $N(x)$ to denote the family of all neighbourhoods of x in X, suppose we arbitrarily form each $N \in N(x)$ an $x_N \in X$. Then the net $(x_n)_N \in N(x)$ converges to x

Converse: Suppose G is not open, then There exists $x \in G$ such that G is not a neighbourhood of x. No neighbourhood of x can lie within G so for all$N \in N(x)$ we can choose $x_N \in N$ $(x_n), n \in N(x) \Rightarrow$ but x_n is never in G

Proposition 5 If in a space $(X, \mathcal{T})$, p is an element and A is a subset then $p \in \bar{A}$ iff there exists a net of all elements belonging to A that converges to p

Lemma 4 For $A \in X$ and $p \in X$, we have that $p \in A$ iff every neighbourhood of p meets A.

Converse: Suppose there is a net x_g goes to p. If N is a neighbourhood of p then $x_g \in N$ for large g and N intersection $A \neq$. Then by the previous lemma $p \in \bar{A}$

Proposition 6 Consider a mapping $f : (X, \mathcal{T}) \mapsto (Y, \mathcal{T})$, f is continuous iff when x_g goes to l in X, then $f(x_g)$ goes to f(l) in Y

Converse: Suppose f is not continuous, then There exists $A \subseteq X$ such that $f(\bar{A}) \subsetneq f(\bar{A})$ There exists $p \in \bar{A}$ such that $f(p) \notin f(\bar{A})$ From the previous proposition we know there exists a net $(a_g)_g \in D$ 0 in A converging to p. But if $f(a_g)$ goes to f(p) then $f(p) \in f(\bar{A})$

4 Conclusion

We have seen the usefulness of nets and now have a very useful tool for tackling convergence in a more abstract sense, as well as a new topology, which can be initially confusing, but rewarding to explore. These methods could be used to explore the convergence of a peano sequence with more time, and more paper.

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USING PERSISTENT HOMOLOGY TO UNDERSTAND POINT CLOUD DATA SETS

James Truman, Ellen Bennett and Enda O Connor

Abstract. In a recently published book, "Topological Data Analysis for Genomics and Evolution", by Rabadan and Blumberg, one of the many new applications of topology in the field of data science was demonstrated. While searching for clusters and cluster reduction algorithms have been used to analyse data for some time, they often fail. A case must be made for different algorithms that capture this higher dimensional shape that data can have. This paper will start by introducing some concepts and definitions from algebraic topology; continuing on to use these tools from algebraic topology and apply them to data analysis; and finally reaching the main point of this paper which is to discuss how persistent homologies can be used to infer meaningful information from point cloud data sets.

1. Introduction to Algebraic Topology

Before starting with Topological data analytics some definitions and ideas from algebraic topology need to be introduced to give the required tools. Algebraic topology uses tools from abstract algebra to study topological spaces, with a goal of finding algebraic invariants that classify topological spaces up to homeomorphism, but most classify up to homotopy equivalence.

1.1. **History.** While Algebraic Topology can be traced back hundreds of years, it was Henri Poincaré (1854-1912) who is regarded as giving topology its "wings". In his revolutionary Analysis Situs (1895 Paper) and subsequent supplementary papers, he provided a systematic approach to the subject developing concepts such as the fundamental group and simplicial homology, and was also the first to coin the term "homeomorphism". Initially developed as a tool to describe global properties of differential equations on surfaces, he concluded that qualitative behaviour of differential equations depended on the shape of their graphed surfaces. The applications of this have come beyond what he could have envisioned in the late nineteenth century, and the last number of years have led to huge applications in TDA.

1.2. **Motives.** Algebraic Topology gives a language and method to express global properties of the shape of geometric objects (some tangible object) and turn them into algebra. Before continuing, some intuitive definitions will be provided for various important terms. An invariant is a property of the object that is not changed by a continuous deformation; and globally means to look at the whole object and not just a portion of it. Examples of algebraic or topological invariants include the number of path components, Betti numbers (proved by Poincare to be

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invariant, it is the max number of cuts that can be made without dividing a surface into separate pieces) or Euler characteristics.[1]

1.3. **Combinatorial Algebraic Topology.** Combinatorial topology is the older name for algebraic topology when all topological properties were expressed, set up and solved in Euclidean space of dimensions 1,2 and 3. There is a long tradition in algebraic topology of studying combinatorial models (description of a space using only discrete data) of geometric objects. The most important of them for us is the simplicial complex.[1]

Definition 1. [4]*A finite collection of simplices in* $\mathbb{R}^n$ *is called a simplicial complex if:*
1. Whenever a simplex lies in the collection, then so too do all of its faces.
2. Whenever two simplicies in the collection intersect, they do so in a common face.

A simplicial complex is very robust and is formed by 'gluing' together simplices of different orders. These then can be used to describe a space or data points. A simplicial complex can also be thought of as a higher dimensional representation of a graph (Simplical complex with orders 0 and 1 only). What is convenient about working with these, is that the algebraic invariants of the spaces which they represent can be algorithmically computed directly from the combinitorial descriptor. The main algebraic invariant to a simplicial complex is the Euler Characteristic.

Definition 2. [4]*Given a topological space* X *with triangulation* $h : |k| \mapsto X$
We define the Euler Characteristic of X *to be*

$$\chi(X) = \chi(k).$$

Where k is a simplicial complex.
These can often be thought of very intuitively. For example, the Euler characteristic of a solid triangle is the same as that of a point. As we crush the solid triangle in on itself we are left with a point. When comparing the Euler characteristic between a hollow and solid triangle we detect the only difference to be the hole in the centre of the hollow triangle. These structures can represent data of any dimension and can form connections between data points and give the data a shape. Thus, as long as we can express our data as a simplicial complex we can apply our tools of algebraic topology and make inference to the shape and to the meaning of our data.

2. Topological Data Analysis

2.1. **Introduction.** TDA is the process of studying the "shape" or topological structure of data in order to infer meaningful information. One of the main methods for performing TDA is persistent homology on point cloud data sets (A point cloud is just a set of data points in space.). Persistent homology is basically defined as the process of computing topological features of data such as the number of connected components, holes or Euler characteristics. These are also known as topological invariants (a property preserved under a homeomorphism). The input for the computation of persistent homology is the point cloud data set and the output is a pair of real numbers (birth and death times) that are plotted as a set on a line called the barcode.

2.2. **History.** Topological data analysis is a very new field of mathematics. Persistent homology and an efficient algorithm to calculate them was introduced by Edelsbrunner et al. in 2000 and it is now the flagship tool used in TDA. Carlsson et al. reformulated the initial definition and gave a visualization method known as persistent barcodes which we will describe and use further on.

2.3. **Persistent Homology.** Topologists use persistent homology to try to accurately recover topological invariants of geometric objects from discrete samples. One method for computing topological features of a point cloud data set is by using the Vietoris Rips Complex. The Vietoris Rips Complex is an abstract simplicial complex that can be defined from any metric space (X, ∂_X) and a specified distance ϵ by forming a simplex for every finite set of points that has diameter at most ϵ.

Definition 3. *(Vietoris-Rips complex)*[1]. *Let (X, ∂_X) be a finite metric space and fix $\epsilon > 0$. The Vietoris-Rips complex $VR_\epsilon(X, \partial_X)$is the abstract simplicial complex with*

1. *vertices the points of X, and*
2. *A k-simplex$[v_0, v_1, ..., v_k]$when*
$$\partial_X(v_i, v_j) \leq 2\epsilon \text{ for all } 0 \leq i, j \leq k.$$

Choosing a single ϵ can be very problematic. There might be distinct feature scales at which meaningful information can be recovered and other feature scales that will just show noise. Topological invariants of the simplicial complex are very unstable, small amounts of noise can cause large changes in the simplicial complex and its homology. The way topological data analysts overcome this is by simultaneously looking at multiple feature scales (a range of values for ϵ). Homological features of the data are calculated for the multiple feature scales. Homological features that exist for a large range of values for ϵ are likely to reflect an underlying property of the data, and these are known as persistent homological features. Unstable or homological features that do not last for long are usually considered to be noise. The persistent homology of a finite metric can be described using the barcode.

Definition 4. *A <u>barcode</u>* [1] *is a multiset of non-empty intervals of the form $[x, y) \subset R$.*

Each interval on the barcode represents the lifespan of a homological feature. Barcodes are robust to perturbations of the input data and this makes them very useful for analysing the features of data.

2.4. **Application in R.** To demonstrate the use of persistent homology in topological data analysis we will use the R studio package "TDAstats". This package gives us a method for studying the homology of data sets at multiple scales simultaneously. We will use two data sets in R, Example 1 is a set of points uniformly distributed within a unit square in the form of a matrix with 100 rows and 2 columns, the x and y coordinates. In Example 2 our data set is a set of points that form a unit circle in the same matrix form.[3]

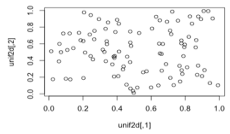

FIGURE 1. Plot

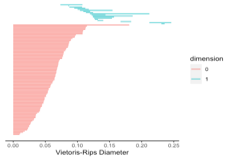

FIGURE 2. Barcode

This is a plot of the first data set, we will use the built in R function to calculate the persistent homology for the data. We then use the plot_barcode function to plot the feature matrix as a barcode.

This red line shows us the birth and death times of 0 cycles and the blue line shows the birth and death times of the 1 cycles. The start of a red or blue line indicates the vietoris-rips diameter when the feature first appears and when the line ends is the diameter when the feature disappears. Long bars on the barcode indicate features while small bars indicate noise. In this example all of the bars are very small none of the bars persist past the diameter 0.25 and therefore there are no homological features in the data.[2]

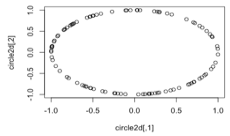

FIGURE 3. Plot

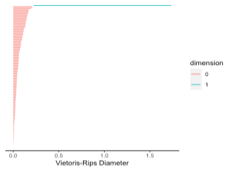

FIGURE 4. Barcode

This is the barcode for the second data set. The long blue bar in the barcode indicates a hole which is the only 1-cycle present in the data, while the short red bars are noise.The single persistence bar is expected as the data forms a unit circle.[2]

2.4.1. *R code used.* library('TDAstats')
data('unif2d')
data('circle2d')
plot(unif2d)
phom.unif ¡- calculate_homology(unif2d)
plot_barcode(phom.unif)
plot(circle2d)
phom.circ ¡- calculate_homology(circle2d)
plot_barcode(phom.circ)[3]

3. Conclusion

In conclusion, we have discussed how concepts from algebraic topology such as invariants, simplicial complexes and homeomorphisms can be used in data analytics. We have also shown how by using statistical software such as R we were able to make the persistant homology calculations and display it graphically using barcodes. To find such a significant application of a pure mathematics subject that has been studied for its own sake for hundreds of years is very exciting and suggests an optimistic future for this rapidly growing field.

> "Pure mathematics, may it never be of any use to anyone"
>
> — Henry John Stephen Smith

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FIXED POINT THEOREM

(Communicated by Artur Zduniak, 14102797, 3FM2)

Abstract. Fixed-point theory originated from successive approximations method. It is currently the most powerful mathematical analysis tool today. The theorem is a beautiful mixture of geometry, analysis, and topology (SpringerOpen, Fixed Point Theory and Applications, n.d.). The theory has numerous applications in Engineering, physics, chemistry game theory, and optimization theory (Vasile Berinde et al., n.d.). In its assertion, the fixed-point theory states under specified conditions, a mapping T of X into itself assumes one or more fixed points. The original publication of the Brouwer's theorem was in 1912 where it was revered to as the H. Poincare equivalent in the 19th century (Karakula, et al., 2018).The fixed-point theorem comprises of other theorems like Contracting Mapping Theorem, Brouwers' Theorem, Newton Raphson Method, and Banach fixed-point theorem. These fixed-point theory tools prove the existence and outstanding nature of solutions in various mathematical models like differential, integral, ordinary, and partial differential equations among others. This paper seeks to confirm that the Fixed-Point Theory is an interdisciplinary topic that is applicable in numerous mathematics disciplines and sciences (Vandana Garg, 2017).

1. Brief History

Fixed point theory has its origins in the successive approximation method. According to Karakula, Rao and Mankena (2018), fixed point theory is the most powerful mathematical analysis tool today. The method was used to prove the existence of differential equations solutions. The fixed-point theory was introduced by two independent mathematicians on 1837 by Joseph Liouville and in 1890 by Charles Emile Picard. However, the formal appearance of the fixed-point theory was in the early twentieth century. According to Pant, Lohani, and Jha (2002), the fixed point theory is a statement asserting that under some conditions, a mapping T of X into itself assumes one or more fixed points. The major classical outcome in fixed-point theory is the work of L.E.J Brouwer. Brouwer's theorem states that for each continuous self-mapping of closed unit in R^n, the n-dimensional Euclidean space has a fixed point. The original publication of the Brouwer's theorem was in 1912 where it was revered to as the H. Poincare equivalent in the 19th century (Karakula, et al., 2018). In 1896, Poincare proved the result that If f: $E_n \to E_n$ is a continuous function with a property r>0 and say $\alpha > 0$, $f(x) + \alpha x \neq 0, ||x|| = r$, then there exists a point x_0, $||x_0||$, $\leq$ r in a way that $f(x_0) = x_0$ (Pant, et al., 2002). This concept forms the Brouwer's fixed point theorem. Further, P. Bohl independently discovered the concept in 1904. A Cauchy in 1844 was the first to prove the outstanding nature and existence of the differential equations solution $\frac{dy}{dx} = f(x, y); y(x_0) = y_0$ where f is a function that is continuously differentiable. In

2010 *Mathematics Subject Classification.* Primary .

1877, R Lipschitz simplified the proof by Cauchy using the present day's *Lipschitz, Condition* (Pant, et al., 2002). Latter G Peano later propagated the supposed of the F's continuity in 1890. Peano's approach is today more closely linked to the fixed-point theory and is used to demonstrate the theory's existence. Using his theorem, Brouwer proved the fixed-point theorem for a square, n-dimensional counter, and a sphere. Kakutani further extended the theory. Later, the Banach principle was invented that became fundamental in the area of functional analysis. Some application areas are game theory, optimization theory, variational inequalities, and approximation theory (Karakula, et al., 2018). Another crucial contribution that plays an intense role in the Fixed-Point Theory is E. Sperner's work in 1928. Sperner proved the combinational geometric Lemma on triangle decomposition. According to Pant, Lohan, & Jha (2002), the Fixed Point Theory has numerous applications and attracts numerous others. The most crucial applications of the theory to be discussed in this paper are the Contracting Mapping Theorem, Brouwers' Theorem, Newton Raphson Method, and Banach fixed-point theorem. Together with others, these fixed-point theory tools prove the existence and outstanding nature of solutions in various mathematical models like differential, integral, ordinary, and partial differential equations among others. Other areas of application are distribution of temperatures, reactions of chemicals, economic theory, steady state temperature distribution, and fluids flow, among others.

2. Applications of Fixed-Point Theory

2.1. **Contracting Mapping Theorem.** The contracting mapping theorem is also called the Banach's fixed point theorem (Wikipedia Banach fixed-point theorem, 2020). The theory is one of the simplest and most crucial methods for construction of solutions of non-linear and linear equations. Where (X, d) is a metric space, the mapping T: $X \to X$ is a contraction mapping provided there exists a constant c, with $0 \leq c < 1$ in that d(T(x),T(y)) $\leq$ cd(x,y) For all x, y ϵ X (Wikipedia, Contraction mapping, 2019).

The theory states that a restricted contraction on a complete metric space has an outstanding fixed point (Pant, et al., 2002). The contraction-mapping theorem has numerous uses in analysis. The applications spread both in the applied and pure fields. The main application area is the Picard's theorem that is the basis of existence and outstanding theorem for ordinary differential equations and the Gauss Seidel method of computing systems of linear equations in numerical analysis (Conrad, 2018). Finally, the Contraction mapping theorem is used in the Google's Page Rank Algorithm:

The two other applications are dynamic optimization and differential equation (Choudhuri, 2017). The dynamic optimization:

$$\max_{(x^m)} \varphi\left(x^0, x^1\right) + \sum_{i=1}^{\infty} \delta^i \varphi\left(x^i, x^{i+1}\right)$$

Such that

$$x^1 \in \Gamma\left(x^0\right) \quad \text{and} \quad x^{m+1} \in \Gamma\left(x^m\right) \text{ for } m = 1, 2, \ldots$$

In the differential equations, the formulae used is (Conrad, 2018)

$$y' = e^{-x^2}$$

and the solution is given by

$$y = \int e^{-x^2} dx$$

2.2. **Brouwers' Theorem.** This prove of this theory occurred in the 1912. The application of this theory is little in the functional analysis area where the concern is more on infinite dimensional subsets of some functional spaces. There are numerous proofs of the existence of Brouwer's theorem. This theory in topology is named after L.E. Brouwer (Wikipedia, Brouwer fixed-point theorem, 2020). The theory states that any continuous function f mapping a compact convex set to itself has a point x0 in such a way that $f(x_0) = x_0$. In the simplest form, the Brouwer fixed point theorem for continuous functions f from a closed interval I in the real numbers to itself or from a closed disk D to itself (Choudhuri, 2017). The most general form of Brouwer's fixed point theorem uses the Euclidean space. The applications of the Brouwer's theorem are in proving deep results in differential equations and contributes to other theorems including the Hairy Ball Theorem, Jordan Curve Theorem, and Borsuk-Ulam Theorem. The Brouwer's theorem is also extended to form the Kakutani fixed-point theorem that proves the general equilibrium existence in market economies (Wikipedia, Brouwer fixed-point theorem, 2020).

2.3. **Newton Raphson Method.** The Newton's method is an algorithm for finding roots. This method produces successfully better applications of the roots or zeroes of a real-valued function (Wikipedia, Newton-Raphson, 2020). Newton's method's basic form involves a single-variable function f whose definition stands for a real variable x, the function's derivative f', and the first guess x_0 for the root of f. The equation is

$$x_1 = x_0 - \frac{f(x_n)}{f'(x_n)}$$

The equation is a better approximation of root compared to x_0.

The Newton's method is applicable in the minimization and maximization problems; solving transcendental equations; multiplicative inverses of numbers and power series; obtaining zeroes of special functions; and numerical verification for solutions of non-linear equations. The real-life application of the Newton's method is in solving flow problems like water distribution flow and the electric power flow. In power system, Newton Method is used for fault analysis, coordination of relays, and contingency among others (Hajer Jmii, 2018).

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