Recall A function $f: X \to Y$ between topological spaces is continuous if every open set $U \subseteq Y$ has open pre-image $f^{-1}(U) \subseteq X$.

~~Cor~~ Example Consider

$$X = \{a, b, c\} \quad T_X = \{ \emptyset, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,b,c\} \}$$

$$Y = \{a, b, c, d\}, \quad T_Y = \{ \emptyset, \{a\}, \{a,b\}, \{a,b,c\}, \{a,b,c,d\} \}.$$

$f: X \to Y$, $\quad f(a) = a$

$\quad f(b) = b$

$\quad f(c) = c$

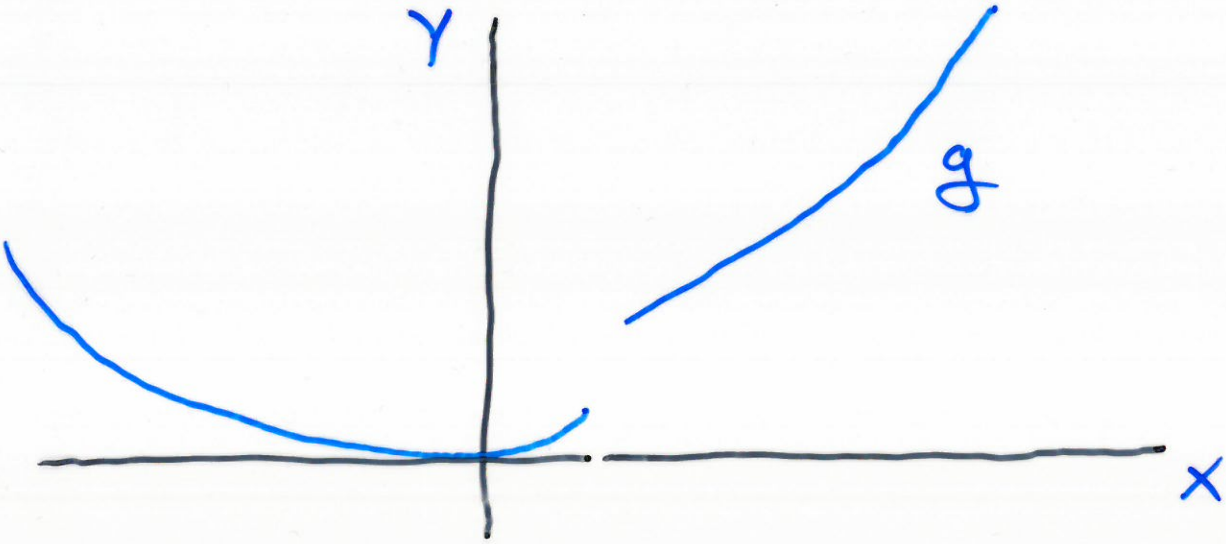
This function is not continuous because $\{a\}$ is open in Y and $f^{-1}\{a\} = \{a\}$ is not open in X.

<u>Example</u> $X = (-\infty, 1) \cup (1, \infty)$

$Y = \mathbb{R}$

Define $g : X \to Y$ by

$$g(x) = \begin{cases} x^2, & x < 1 \\ x^2 + 1, & x > 1 \end{cases}$$



This is continuous.

Major definition: A continuous function $f: X \to Y$ between topological spaces is a **homeomorphism** if there exists a continuous function $g: Y \to X$ such that

$$g(f(x)) = x \quad \text{for all } x \in X$$

and

$$f(g(y)) = y \quad \text{for all } y \in Y.$$

Defn Two topological spaces X and Y are **homeomorphic** if there exists some homeomorphism $f: X \to Y$.

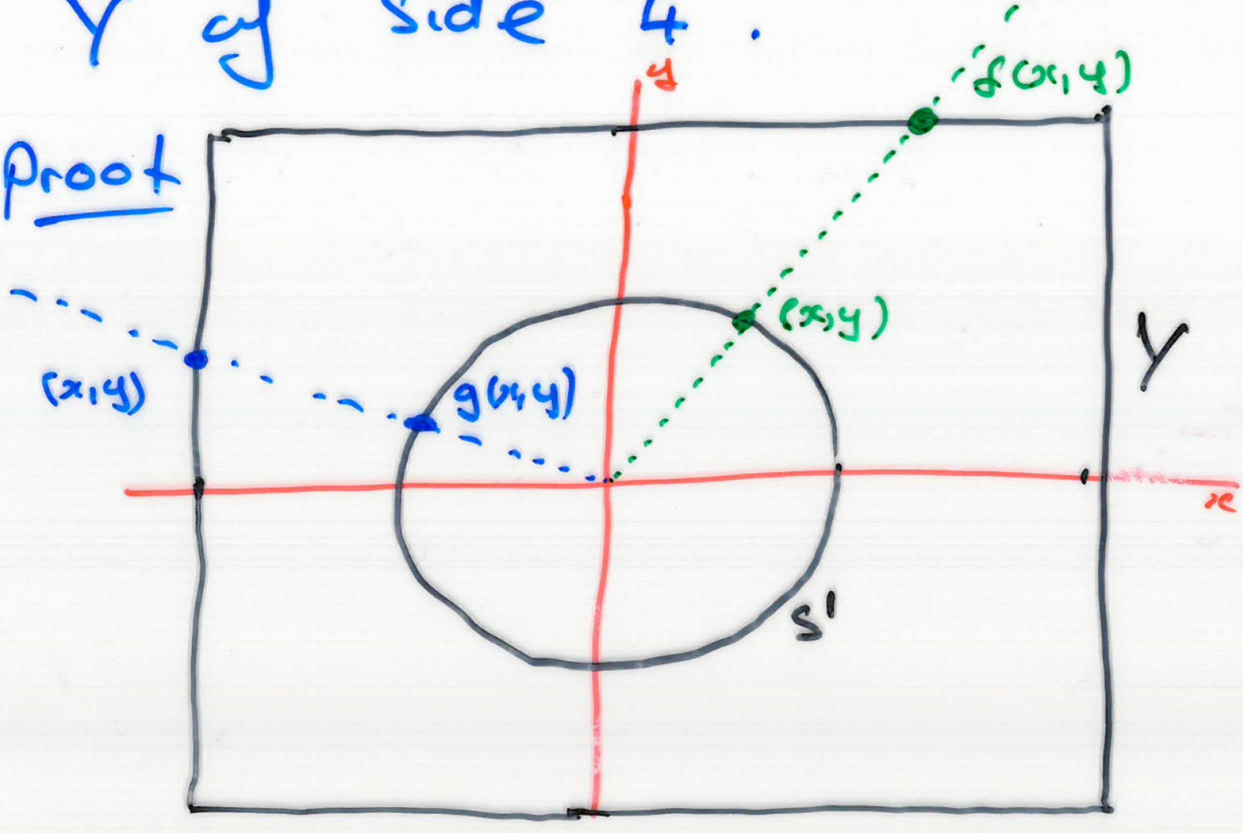
Defn A property is said to be <u>topological</u> if, whenever some space X has this property, then so too do all spaces Y which are homeomorphic to X.

Example The unit circle

$X = S^1 = \{ (x,y) \in \mathbb{E}^2 : x^2+y^2 = 1 \}$

is homeomorphic to the square Y of side 4.

Proof



Consider $f: S^1 \to Y$, $(x,y) \mapsto f(x,y)$

where $f(x,y)$ is the intersection

with Y of the ray ~~through~~ from

the origin through (x,y).

Consider $g: Y \to S^1$, $(x,y) \mapsto g(x,y)$

where $g(x,y)$ is the intersection

with S^1 of the ray from

the origin through (x,y).

Note: Both f and g are continuous

and

$$g(f(x,y)) = (x,y)$$

and

$$f(g(x,y)) = (x,y).$$

Hence the square and the circle are homeomorphic.

$\boxed{M}$

Proposition If $f: X \to Y$ and $g: Y \to Z$ are continuous functions of topological spaces, then their composite

$$g \circ f : X \to Z \qquad x \mapsto g(f(x))$$

is continuous.

Proof Let $U \subseteq Z$ be an open subset in Z. Then $g^{-1}(U)$ is open in Y since g is continuous. Then $f^{-1}(g^{-1}(U))$ is open in X since f is continuous.

But
$$(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)).$$

$\blacksquare$