Example Let $X = \mathbb{R}^n$. Let T consist of just two open sets

$$T = \{ \phi, X \}.$$

Then (X, T) is a topological space.

We call T the trivial topology.

This space is connected.

Defn Let X be a set with a topology T. Let $Y \subseteq X$ be a subset of X.

In the subspace topology on Y a subset $U \subseteq Y$ is deemed to be open if and only if

$$U = Y \cap A$$

with A an open set in X.

with this topology we call Y a <u>topological subspace</u> of X.

<u>Example</u> $X = \mathbb{R}$ with standard topology in which a subset $U \subseteq \mathbb{R}$ is open if, for any $x \in U$, there is an $\varepsilon > 0$ with

$$(x - \varepsilon, x + \varepsilon) \subseteq U.$$

Consider the integers $\mathbb{Z} \subseteq \mathbb{R}$, with subspace topology. Then $\mathbb{Z}$ is <u>not</u> connected because

$$U = \{ n \in \mathbb{Z} : n \geq 0 \}$$

$$V = \{ n \in \mathbb{Z} : n < 0 \}.$$

Clearly $U \cup V = \mathbb{Z}$, $U \cap V = \emptyset$.

And U, V are open in $\mathbb{Z}$ because

$$U = \mathbb{Z} \cap (-\tfrac{1}{2}, \infty)$$

$$V = \mathbb{Z} \cap (-\infty, -\tfrac{1}{2})$$

Definition A connected component of a topological space X is a connected subspace $Y \subseteq X$ such that there is no connected subspace $W \subseteq X$ with $Y \subsetneq W$.

Example

Let $X = \{ (x,y) \in \mathbb{E}^2 : x^2+y^2 \neq 1 \}$.

There are two connected components of X, namely

$$Y = \{ (x,y) \in \mathbb{E}^2 : x^2+y^2 < 1 \}$$

and

$$Z = \{ (x,y) \in \mathbb{E}^2 : x^2+y^2 > 1 \}.$$

Example For the real line $\mathbb{E}^1$ we have the subset $\mathbb{Q} \subseteq \mathbb{E}^1$ of rational numbers.

with the subspace topology on $\mathbb{Q}$ we see that $\mathbb{Q}$ is <u>not</u> connected. For instance

$$A = \{ x \in \mathbb{R} : x > \sqrt{2} \} = (\sqrt{2}, \infty)$$

$$B = \{ x \in \mathbb{R} : x < \sqrt{2} \} = (-\infty, \sqrt{2})$$

Then

$$\mathbb{Q} = (A \cap \mathbb{Q}) \cup (B \cap \mathbb{Q}).$$

So $A \cap \mathbb{Q}$ and $B \cap \mathbb{Q}$ are both open in $\mathbb{Q}$. And

$$(A \cap \mathbb{Q}) \cap (B \cap \mathbb{Q}) = \emptyset.$$

The connected components of $\mathbb{Q}$ are the subspaces

$$\{x\}$$

for each $x \in \mathbb{Q}$.