

## Definition (Riesz [1909], Hausdorff [1914])

A topological space consists of a set  $X$  and a collection  $T$  of subsets of  $X$  which we call open. The following axioms must hold:

- T1) The union of any collection of open sets is open.
- T2) The intersection of any finite collection of open sets is open.
- T3) Both  $\emptyset$  and  $X$  are open.

Example 1  $X = \{1, 2, 3, 4\}$

$$T = \left\{ \emptyset, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}, \{1, 2, 3, 4\} \right\}$$

this is a topological space.

Example 2  $X = \{1, 2, 3, 4\}$

$\tau = \{\emptyset, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}, \{1, 2, 3, 4\}\}$

Not a topological space.  $T_2$  fails

Example 3

$X = \{1, 2, 3, 4\}$

$\tau = \{\emptyset, \{2\}, \{3\}, \{1, 2, 3\}, \{1, 2, 3, 4\}\}$

Not a topology because  $T_1$  fails

Example 4

$X = \{1, 2, 3, 4\}$

$\tau = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 3, 4\}\}$

This is a topological space.

Example 5 Let  $X = \mathbb{Z}$ . The

cofinite topology on  $X$  has,

as open sets, those subsets

$U \subseteq X$  such that the complement  $X \setminus U$

is finite. Also, we deem  $\emptyset$

to be open. This is a topological space.

For the next example we need some notation.

For  $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$  we define the Euclidean norm

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$$

For  $x, y \in \mathbb{R}^n$  we define the Euclidean distance

$$d(x, y) = \|x - y\|$$

We write  $\mathbb{E}^n$  to denote the set  $\mathbb{R}^n$  endowed with this Euclidean distance.

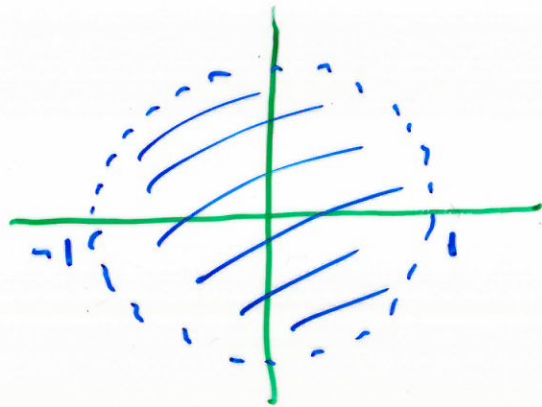
For  $x \in \mathbb{E}^n$  and for any real number  $\varepsilon > 0$  we define the open ball central at  $x$  of radius  $\varepsilon$

$\varepsilon$  to be

$$B^n(x, \varepsilon) = \{y \in \mathbb{E}^n : d(x, y) < \varepsilon\}$$

Examples

$$B^2((0,0), 1)$$



$$B^1(0, 1) \\ = (-1, 1)$$



Example  $X = \mathbb{R}^n$

Let  $\mathcal{T}$  consist of those subsets  $U \subseteq \mathbb{R}^n$  such that, for any  $x \in U$  we can find  $\varepsilon > 0$  such that the  $\varepsilon$ -Euclidean ball

$$B^n(x, \varepsilon)$$

lies entirely within  $U$ :

$$B^n(x, \varepsilon) \subseteq U$$

This defines a topological space.

Definition A topological space  $X$  is said to be connected if it can not be expressed as a union

$$X = U \cup V$$

where  $U, V$  are non-empty open subsets of  $X$ , and where  $U \cap V = \emptyset$ .

Example 1 is connected.

Example 2 are not topological spaces so it doesn't make sense to ask if they are connected.

Example 5  $\mathbb{Z}$  is connected with the cofinite topology.

Example 6  $\mathbb{E}^n$  is connected.

Example 7 Let  $X = \mathbb{R}^n$ . Let  $\tau$  consist of all subsets of  $X$ . Then  $(X, \tau)$  is a topological space. We call  $\tau$  the discrete topology.

This space is not connected.

For instance

$$U = \{ (x_1, \dots, x_n) \in \mathbb{R}^n : x_1 > 0 \}$$

$$V = \{ (x_1, \dots, x_n) \in \mathbb{R}^n : x_1 \leq 0 \}$$

Then  $U \cup V = X$ ,  $U \cap V = \emptyset$ ,

$$U \neq \emptyset \neq V.$$