Last lecture:

$\pi_1(S^1, 1) \cong \mathbb{Z}$

<u>Fundamental theorem of algebra</u>

Any polynomial

$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$

with $a_i \in \mathbb{C}$ and of degree $n > 0$

has at least one zero in $\mathbb{C}$.

<u>Proof</u> Since $a_n \neq 0$, a scalar multiple of the polynomial has the form

$p(z) = z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \cdots + a_0$.

Let's suppose $p(z) \neq 0$ for all $z \in \mathbb{C}$.

For $\lambda \geq 0$ define

$$f_\lambda : S^1 \longrightarrow S^1$$

by

$$f_\lambda(z) = \frac{p(\lambda z)}{|p(\lambda z)|}.$$

Any two of these maps $f_\lambda, f_{\lambda'}$ are homotopic via the homotopy

$$H_t(z) = \frac{p(((1-t)\lambda + t\lambda')z)}{|p(((1-t)\lambda + t\lambda')z)|}$$

Note: For $\lambda = 0$ we have that f_0 is a constant function, and has winding number 0.

Exercise (tricky)

For large λ we have that

$f_\lambda(z)$ is homotopic to

$$g_n : S^1 \to S^1, \quad z \mapsto z^n .$$

But g_n has winding number n.

But $g_n \simeq f_\lambda \simeq f_0$, and homotopic maps have the same winding number.

Hence the winding number of g_n is 0.

Contradiction, $n \neq 0$. $\boxed{}$

Game Theory

A **game** involves

- n players
- a set S_i of strategies for player i.
- a payoff function $v_i : S_1 \times S_2 \times \cdots \times S_n \longrightarrow \mathbb{R}$ for each player i, $i = 1, 2, \ldots, n$.

Example 1 Two player, Mary and John. They want to go to the cinema (c) or to a soccer match (s) together. $S_1 = \{c, s\}$, $S_2 = \{c, s\}$.
Mary $i=1$ John $i=2$

$v_1(c,c) = 2$ $v_2(c,c) = 1$	$v_1(c,s) = 0$ $v_2(c,s) = 0$
$v_1(s,c) = 0$ $v_2(s,c) = 0$	$v_1(s,s) = 1$ $v_2(s,s) = 2$

Example 2 Two players, each places a coin on the table. Player 1 wants coins to be the same. Player 2 wants the coins to be different.

$$S_1 = \{H, T\}$$

$$S_2 = \{H, T\}$$

Payoff function:

$v_1(H,H) = 1$ $v_2(H,H) = -1$	$v_1(H,T) = -1$ $v_2(H,T) = 1$
$v_1(T,H) = -1$ $v_2(T,H) = 1$	$v_1(T,T) = 1$ $v_2(T,T) = -1$

In a pure strategy game each player decides before the game on a strategy to play.

A pure Nash equilibrium occurs if, having played the game, no player benefits from unilaterally changing his/her choice of strategy.

Example 1 There are two Pure Nash Equilibria:

- both go to Cinema
- both go to Soccer.

Example 2 There is no pure Nash equilibrium.