Recall

$$D^n = \{ x \in \mathbb{R}^n : \|x\| \leq 1 \}$$

Brouwer's Theorem

For any continuous function

$$f : D^n \to D^n$$

there exists at least one $x \in D^n$ such that $f(x) = x$.

Defn If $f(x) = x$ we say that x is a fixed point of the function f.

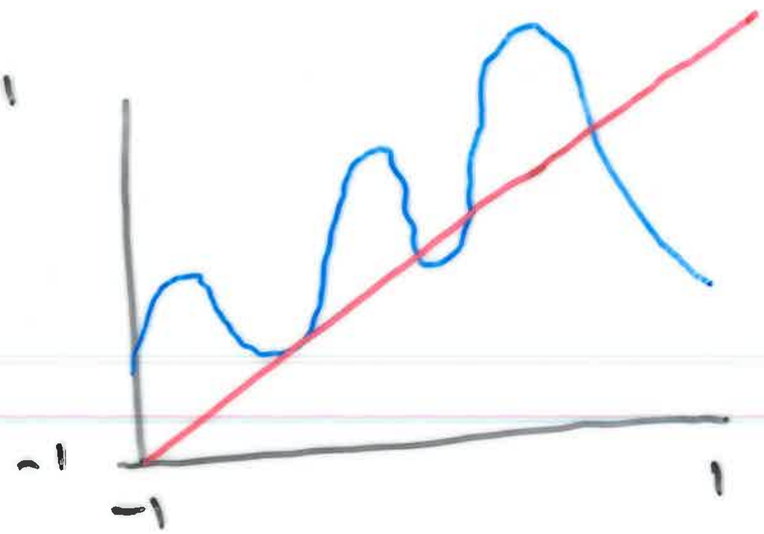
Case $n = 1$

$D^1 = [-1, 1]$

we picture a map

$$f : [-1, 1] \longrightarrow [-1, 1]$$

by its graph



A fixed point is a point where the blue graph of $f(x)$ intersects the red line $y = x$.

Proof of Brouwer's Theorem

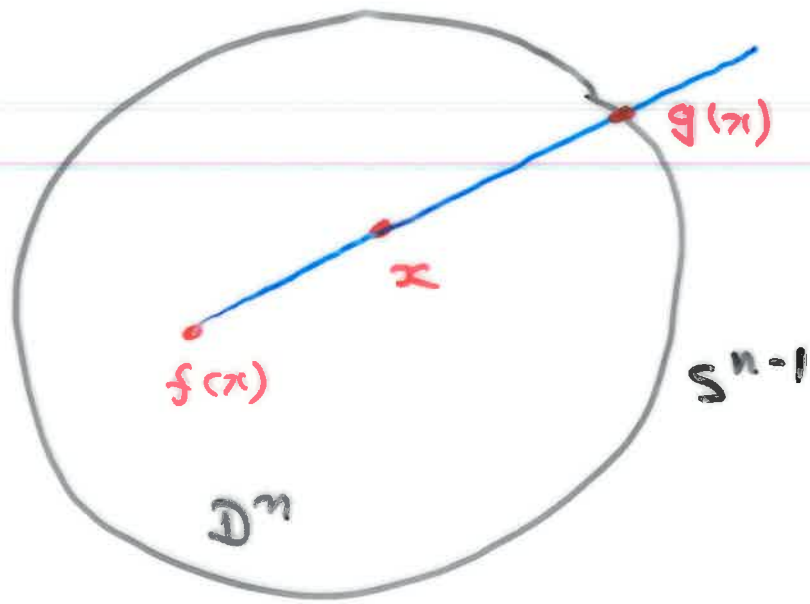
Let $f: D^n \to D^n$ be continuous.

Suppose f has no fixed point.

Then we could define a

continuous map

$$g: D^n \longrightarrow S^{n-1}, \quad x \longmapsto g(x)$$



where $g(x)$ is the point in S^{n-1}

where the ray from $f(x)$

through x intersects S^{n-1}.

Note that $g(x)$ is continuous.

Let $h : S^{n-1} \longrightarrow D^n$, $x \longmapsto x$

Now

$$gh : S^{n-1} \longrightarrow S^{n-1}$$

is clearly the identity on S^{n-1}.

Now $hg : D^n \longrightarrow D^n$ is homotopic to the identity on D^n, via the homotopy

$$H(x,t) = x + t(g(x) - x)$$

$$H(x,0) = x$$
$$H(x,1) = g(x).$$

So $hg \simeq 1_{D^n}$

$$gh \simeq 1_{S^{n-1}}$$

Therefore D^n is homotopy equivalent to S^{n-1}.

Then our major theorem implies

$$\chi(D^n) = \chi(S^{n-1})$$

But

$$D^n \simeq \{0\}$$

and so $\chi(D^n) = 1$.

Last lecture:

$$\chi(S^{n-1}) = \begin{cases} 0, & \text{even} \\ 2, & \text{odd}, \end{cases}$$

Contradiction!

Hence f must have a fixed point. $\boxed{\text{M}}$

Theorem (Frobenius-Perron)

Let A be a real $n \times n$ matrix with entries $a_{ij} > 0$ for all i, j. Then A has a positive eigenvalue. Moreover there is a corresponding eigenvector $v = (x_1, x_2, \ldots, x_n)$ with $x_i \geq 0$ all i.

Proof

Define $\sigma : \mathbb{R}^n \to \mathbb{R}, (x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i$.

$$\Delta^{n-1} = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n : \begin{array}{l} x_i \geq 0 \\ \text{and } \sum_{i=1}^{n} x_i = 1 \end{array} \right\}.$$

Define

$$g : \Delta^{n-1} \to \Delta^{n-1}, \quad x \longmapsto \frac{1}{\sigma(Ax)} Ax$$

Now g is continuous, and

Δ^{n-1} is (homeomorphic to)

D^{n-1}.

Brouwer's Theorem says that

g has at least one fixed

point.

$$x = g(x) = \frac{1}{\sigma(Ax)} Ax .$$

so $Ax = \sigma(Ax) . x$

Therefore x is an eigenvector

of g with eigenvalue

$\lambda = \sigma(Ax) > 0.$ $\boxed{\text{QED}}$