Third Class Test: Wed 27 March

Two maps

$$f, g : X \longrightarrow Y$$

are homotopic if there exist a map

$$H : X \times [0,1] \longrightarrow Y, \quad (x,t) \longmapsto H_t(x)$$

such that

$$H_0(x) = f(x),$$

$$H_1(x) = g(x).$$

Defn Two topological spaces X, Y are homotopy equivalent if there exist maps

$$f : X \to Y, \quad g : Y \to X$$

with

$$fg \simeq 1_Y \quad \text{and} \quad gf \simeq 1_X$$

where $\simeq$ means homotopic,
and $1_X : X \to X$ is the identity function on X, $1_Y : Y \to Y$ is the identity on Y.

<u>Example 1</u> If X and Y are homeomorphic then they are also homotopy equivalent.

If X and Y are homeomorphic there exist maps. $f : X \to Y$, $g : Y \to X$ with

$$fg = 1_Y, \quad gf = 1_X .$$

(Reflexive property of the equiv. relation.)

Example 2 $\quad X = \mathbb{C} \setminus \{0\}$

$$Y = S^1 = \{ z \in \mathbb{C} : |z| = 1 \}.$$

The spaces X and Y are homotopy equivalent, since we have

$$g : Y = S^1 \longrightarrow X, \quad z \longmapsto z$$

$$f : X \longrightarrow Y = S^1, \quad z \longmapsto \frac{1}{|z|} z$$

Clearly

$$fg(z) = z, \qquad fg = 1_Y$$

$$gf(z) = \frac{1}{|z|} z, \quad gf \simeq 1_X.$$

To see that $gf \simeq 1_X$ we use the homotopy

$$H : X \times [0,1] \to X, \quad (z, t) \mapsto \left(\frac{1-t}{|z|} + t\right) z$$

and note

$$H_0(z) = \frac{z}{|z|} = gf(z)$$

$$H_1(z) = z = 1_X(z).$$

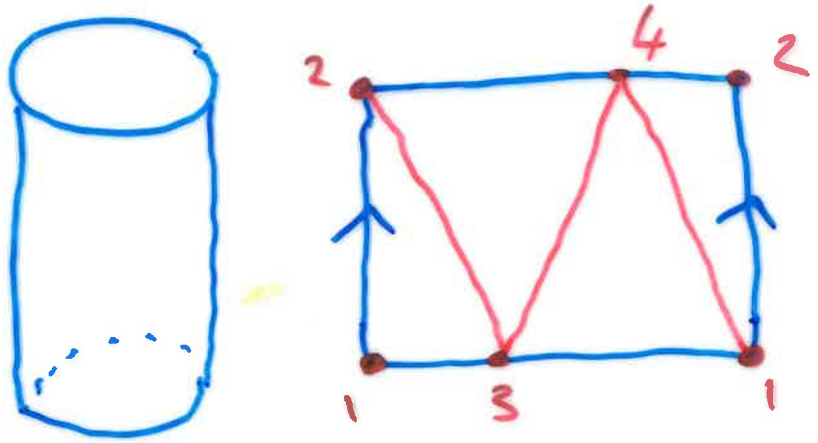
<u>Major</u> <u>Theorem</u>

Let X, Y be spaces with triangulations. If X and Y are homotopy equivalent then

$$\chi(X) = \chi(Y).$$

Illustration The cylinder $S^1 \times [0,1]$ is homotopy equivalent to a circle S^1.

$S^1 \times [0,1]$



$$\chi(S^1 \times [0,1]) = 4 - 8 + 4 = 0$$

S^1

$$\chi(S^1) = 3 - 3 + 0 = 0.$$

<u>Illustration</u> Then n-simplex Δ^n is homotopy equivalent to the space $\{*\}$ consisting of a single point.

So $\chi(\Delta^n) = 1$.

<u>Illustration</u>

$$\chi(S^n) = \chi(\Delta^{n+1}) \pm 1 = 1 \pm 1$$

$$= 2 \text{ or } 0.$$

In fact:

$$\chi(S^n) = \begin{cases} 0 & n \text{ odd} \\ 2 & n \text{ even} \end{cases}$$

We'll use the above major theorem to prove:

Brouwer's Theorem

Let Δ^n be the n-simplex.

Any continuous map

$$f : \Delta^n \longrightarrow \Delta^n$$

has a fixed point, i.e. a point $x \in \Delta^n$ such that $f(x) = x$.