Defn A triangulation of a topological space X consists of a simplicial complex K and a homeomorphism

$$h : |K| \longrightarrow X.$$

Example Consider

$$X = S^1 = \{ z \in \mathbb{C} : |z| = 1 \}.$$

Consider the simplicial complex

$K =$ 

There is a homeomorphism

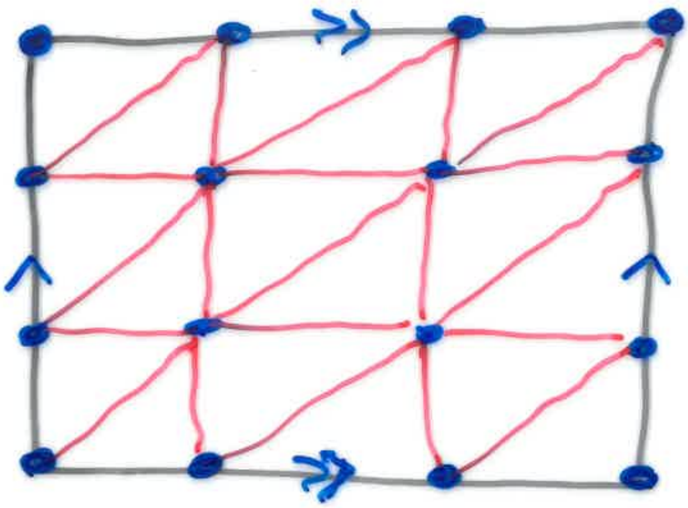
$$h : |K| \longrightarrow S^1.$$

So this is an example of a triangulation of S^1.

Example Triangulation of the torus.

$S^1 \times S^1$

$=$



$\alpha_0 = 9$

$\alpha_1 = 27$

$\alpha_2 = 18$

This describes a simplicial complex K homeomorphic to a torus.

$$\chi(\text{torus}) = \chi(K) = 9 - 27 + 18 = 0.$$

<u>Note</u> For a simplicial K we let α_0 denote the number of vertices

α_1 " " " edges

α_k " " " k-simplexes

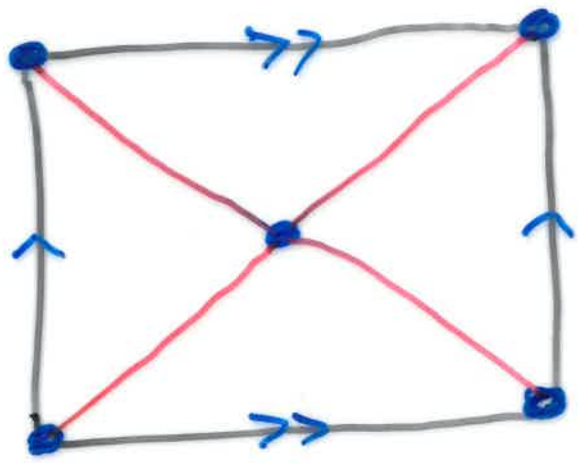
We define the Euler characteristic of a simplicial complex K to be

$$\chi(K) = \alpha_0 - \alpha_1 + \alpha_2 - \alpha_3 + \cdots$$

<u>Defn</u> Given a topological space X with triangulation $h : |K| \to X$, we define the Euler characteristic of X to be

$$\chi(X) = \chi(K).$$

Example (Not a triangulation of the torus)



There is no 1-simplex with just one vertex! There is no triangle with just two vertices.

Theorem (Most profound in the course)

If two simplicial complexes K and L are such that $|K|$ is homeomorphic to $|L|$ then

$$\chi(K) = \chi(L).$$

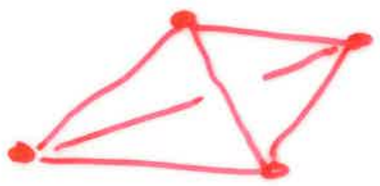
We've seen:

$$\chi(S^1 \times S^1) = 0$$

$$\chi(S^1) = 6 - 6 = 0$$

Example

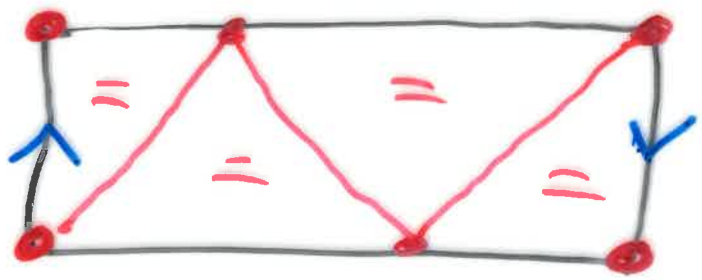
$$\chi(S^2) = 4 - 6 + 4 = 2$$



$\simeq S^2$

Example

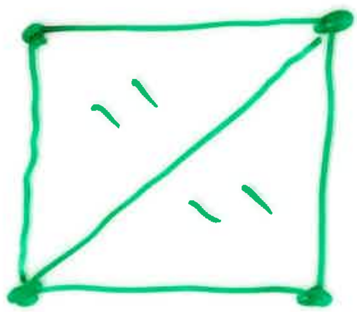
Determine the Euler characteristic of a Möbius band.



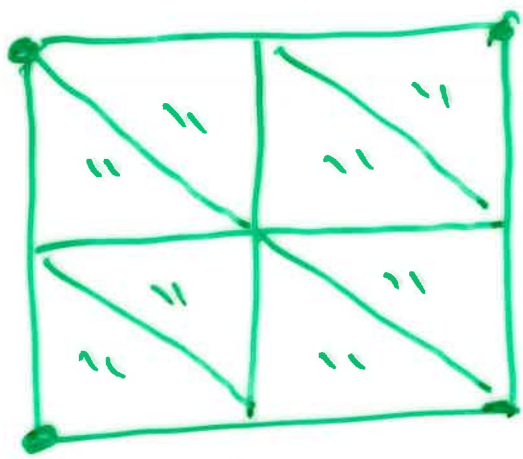
$$\chi(\text{Möbius band}) = 4 - 8 + 4 = 0$$

Early attempts at proving the above theorem focused on:

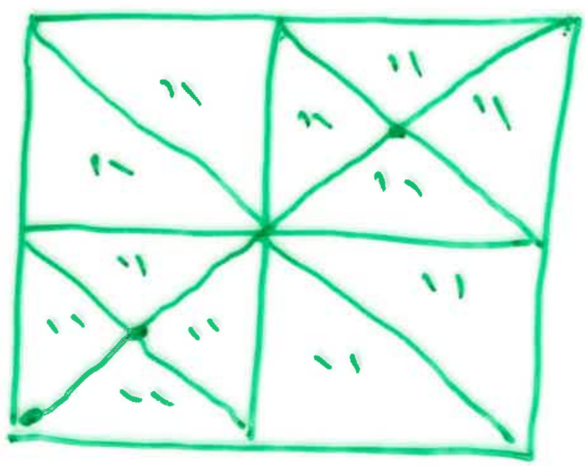
<u>Hauptvermutung</u>: If K and L are triangulations of X then there are "subdivisions" K' of K and L' of L such that $K' = L'$.



K



L



$L' = K'$

The Hauptvermutung was proved for simplicial complexes of dimension ≤ 3 by Morse in the 1950s.

In 1961 John Milnor proved the Hauptvermutung is false in dimensions ≥ 6.