Linear Algebra

There is no linear isomorphism

$$\phi : \mathbb{R} \longrightarrow \mathbb{R}^2$$

Topology

There is no homeomorphism

$$\psi : \mathbb{R} \longrightarrow \mathbb{R}^2$$

(To see this, note that $\mathbb{R} \setminus \{0\}$ is not connected, whereas $\mathbb{R}^2 \setminus \{\psi(0)\}$ <u>is</u> connected.)

Linear algebra

There is no linear surjection

$$\phi : \mathbb{R} \twoheadrightarrow \mathbb{R}^2$$

(To see this

$$\dim(\operatorname{im}(\phi)) + \dim(\ker \phi) = \dim(\mathbb{R})$$

So $\dim(\operatorname{im}(\phi)) \leq 1$

But $\dim(\mathbb{R}^2) = 2$)

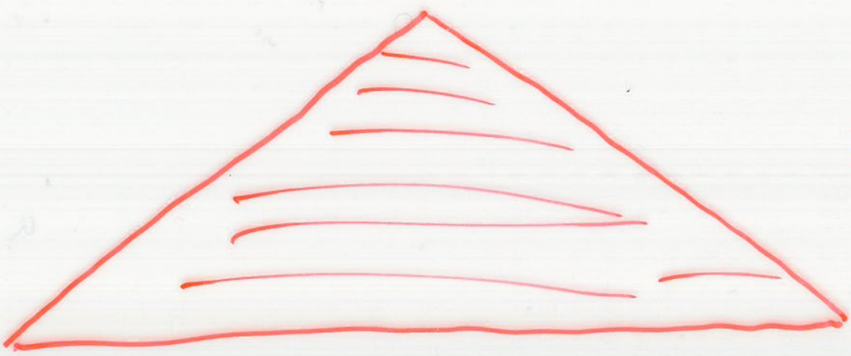
Topology

There exists a surjective continuous function
$\phi : \mathbb{R} \twoheadrightarrow \mathbb{R}^2$

i.e. $\phi(\mathbb{R}) = \mathbb{R}^2$. Such a ϕ is called a space-filling curve.

Aim: we'll construct an example of a surjective continuous map.

Let Δ be an equilateral triangular region of $\mathbb{R}^2 = \mathbb{E}^2$ of side 1.

$\Delta =$  $\subseteq \mathbb{E}^2$

<u>Theorem</u> (Peano) There exists a surjective continuous function

$$f : [0,1] \longrightarrow \Delta$$

<u>Proof</u> We'll first construct a sequence of continuous functions

$$f_1 : [0,1] \longrightarrow \Delta$$

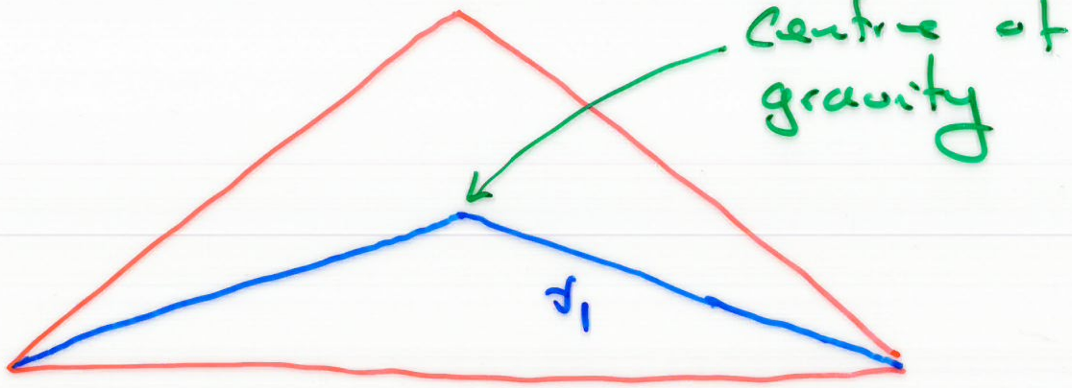
$$f_2 : [0,1] \longrightarrow \Delta$$

$$f_3 : [0,1] \longrightarrow \Delta$$

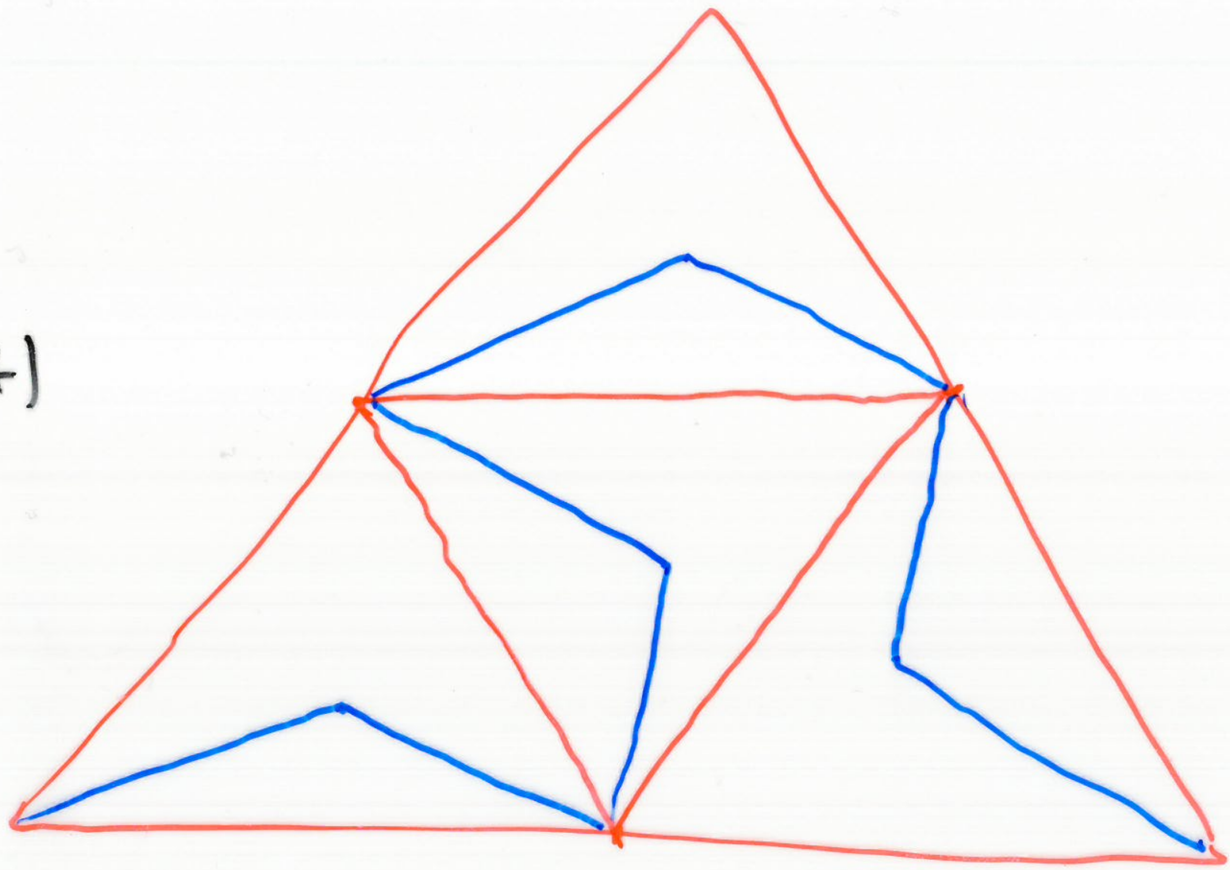
$$\vdots$$

We'll take f to be the "limit" of the sequence, and we'll convince ourselves that this limit is surjective and continuous.

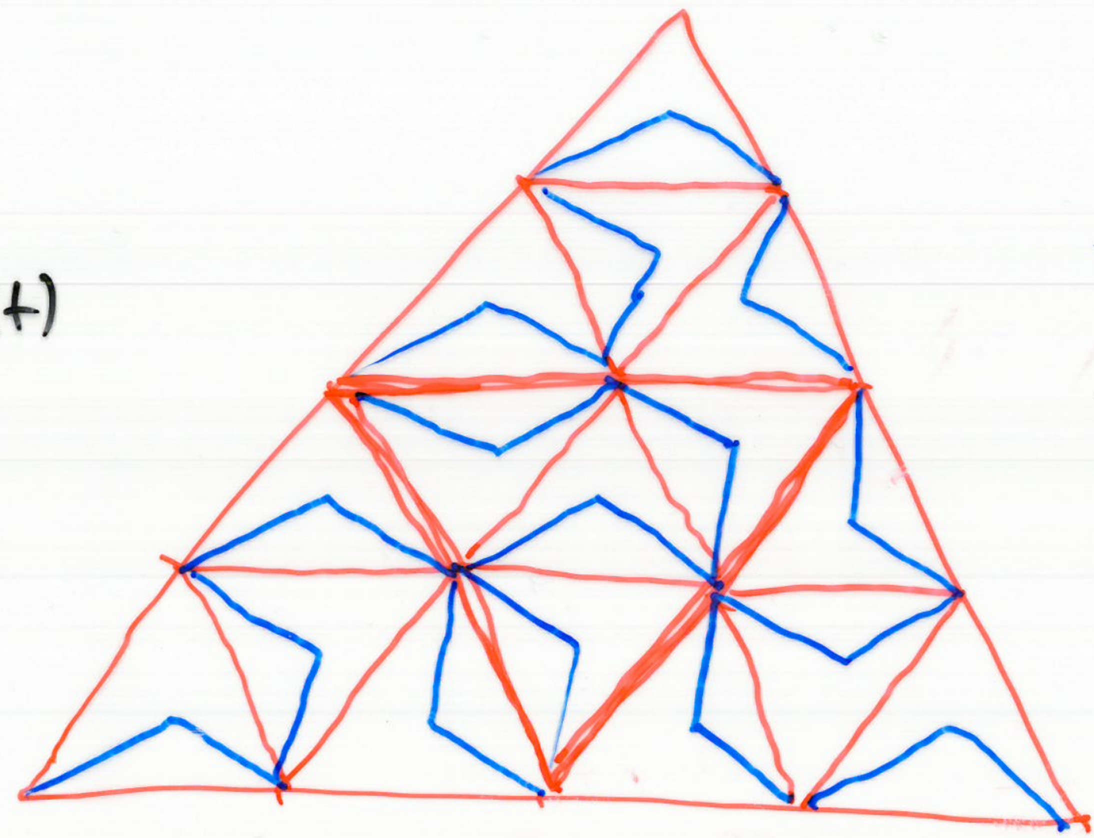
$f_1(t)$



$f_2(t)$



$f_3(t)$



For $f_n : [0,1] \longrightarrow \Delta$ we subdivide Δ into 4^{n-1} red triangles, and the image of f_n inside each triangle looks like a small copy of f_1.

To complete the proof of Peano's theorem we:

1) Define f to be the limit of $f_1, f_2, f_3, \dots$.

2) Need to show that the limit is continuous and onto. (Compactness is used here.)

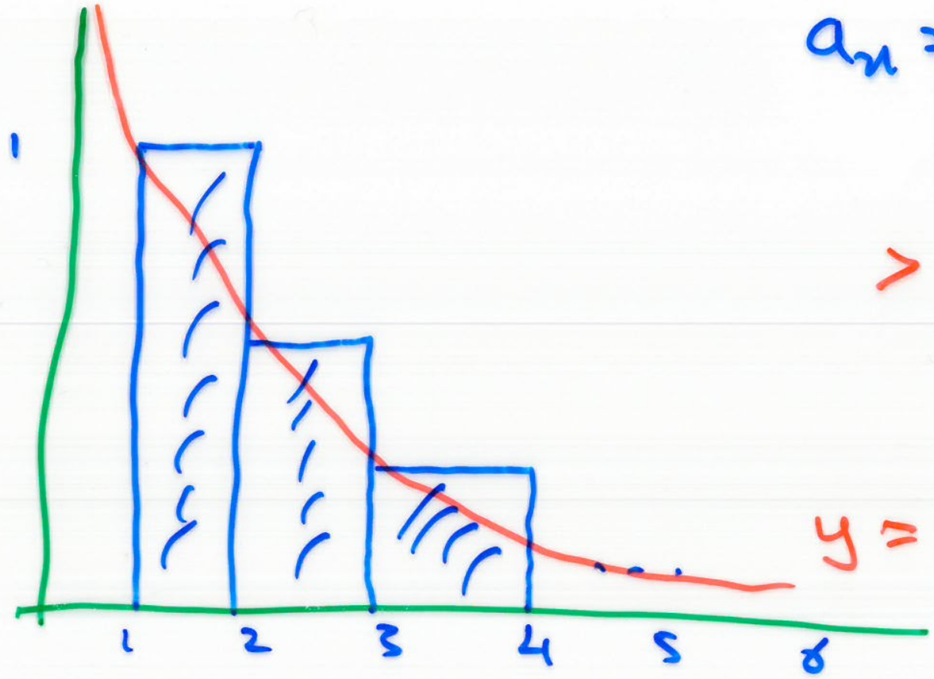
Aside: Some basics on limits

Consider

$a_1 = 1,\ a_2 = 1\frac{1}{2},\ a_3 = 1\frac{5}{6},\ \dots,\ a_n = a_{n-1} + \frac{1}{n}, \dots$

so $\lim_{n\to\infty} a_n - a_{n-1} = \lim_{n\to\infty} \frac{1}{n} = 0$

$\lim_{n\to\infty} a_n$ does not exist.



a_n = area of first n boxes

$> \int_1^{n+1} \frac{1}{x}\,dx$

$= \ln(n+1)$

$y = \frac{1}{x}$

<u>Defn</u> A sequence of points $a_1, a_2, \dots$ in $\mathbb{E}^k$ is said to be a <u>Cauchy sequence</u> if, for $\varepsilon > 0$ there is an N such that

$$\|a_m - a_n\| < \varepsilon$$

for all $m, n > N$.

Theorem Any Cauchy sequence $a_1, a_2, \ldots,$ in $\mathbb{E}^k$ has a limit $\lim_{n \to \infty} a_n$.