We want to say $[0,1]$ is "compact" but that $\mathbb{R}$ is not "compact".

Let X be a topological space. Let $\mathcal{F}$ be a family of open subsets of X whose union equals X. We say that $\mathcal{F}$ is an <u>open</u> <u>cover</u> of X.

<u>Example</u> Let $X = \mathbb{R}$, usual topology

Let $\mathcal{F} = \{ (n-2, n+2) \}_{n \in \mathbb{Z}}$

Then $\mathcal{F}$ is an open cover of $\mathbb{R}$.

Example 2 Let $X = [0,1]$, usual subspace of $\mathbb{R}$ topology.

$$\mathcal{F} = \left\{ [0,\tfrac{1}{3}), (\tfrac{1}{4}, \tfrac{3}{4}), (\tfrac{2}{3}, 1] \right\}.$$

Let $\mathcal{F}$ be some open cover of X. If $\mathcal{F}'$ is a subfamily of $\mathcal{F}$ and if the union of all the sets in $\mathcal{F}'$ equals X, we say that $\mathcal{F}'$ is a <u>subcover</u> of $\mathcal{F}$.

Example 3 $X = \mathbb{R}$

$$\mathcal{F} = \{ (n-2, n+2) \}_{n \in \mathbb{Z}}$$

$$\mathcal{F}' = \{ (n-2, n+2) \}_{n \in 2\mathbb{Z}}$$

Then $\mathcal{F}'$ is a subcover of $\mathcal{F}$ since the union of $\mathcal{F}'$ equals $X = \mathbb{R}$.

An open cover $\mathcal{F}$ of X is said to be finite if $\mathcal{F}$ involves just finitely many sets.

Example 2 is an example of a finite open cover of $X = [0,1]$.

Defn A topological space X is compact if every open cover $\mathcal{F}$ of X has a finite subcover.

Example $\mathbb{R}$, usual topology. By considering $\mathcal{F} = \{(n-2, n+2)\}_{n \in \mathbb{Z}}$ we see that $\mathbb{R}$ is not compact.

The following states that compactness is a topological property.

<u>Proposition</u> Suppose that $f : X \to Y$ is a surjective continuous function. If X is compact then so too is Y.

<u>Proof</u> Suppose X is compact and f is continuous & surjective.

Let $\mathcal{F}$ be some open cover of Y. Then

$$\mathcal{G} = \{f^{-1}(U)\}_{U \in \mathcal{F}}$$

is an open cover of X.

Since X is compact $\mathcal{G}$ has a finit subcover $\mathcal{G}'$.

Let's say

$$\mathcal{G}' = \{ f^{-1}(U) \}_{U \in \mathcal{F}'}$$

with $\mathcal{F}'$ a finite subcollection of $\mathcal{F}$.

But

$$\{ U \}_{U \in \mathcal{F}'}$$

is a finite subcover of $\mathcal{F}$ since the union of $\mathcal{F}'$ must be Y.

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<u>Example</u> $(0,1)$ subspace of $\mathbb{R}$.

This is not compact because it is homeomorphic to $\mathbb{R}$.

Theorem $[0,1]$ is compact.

This theorem proves that $[0,1]$ is <u>not</u> homeomorphic to $(0,1)$.

Proof of theorem

Let $\mathcal{F}$ be any open cover of $[0,1]$.

Define

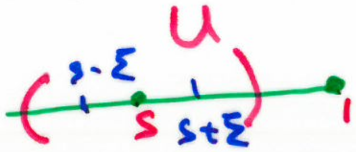
$$X = \{ x \in [0,1] : [0,x] \text{ is contained in the union of a finite subfamily of } \mathcal{F}. \}$$

- X is non-empty since $0 \in X$
- if $0 \leq y \leq x$ and if $x \in X$ then $y \in X$.

• X has a least upper bound s since X is bounded above by 1. (Completeness Axiom for $\mathbb{R}$.)

We need to show that $s = 1$ and $s \in X$, as that will mean that a finite subfamily of $\mathcal{F}$ has union equal to $[0, 1]$.

Suppose $s \neq 1$. Let $U \in \mathcal{F}$ be chosen with $s \in U$.



Choose $\varepsilon > 0$ with $(s-\varepsilon, s+\varepsilon) \subseteq U$. Let $\mathcal{F}'$ be a finite subfamily of $\mathcal{F}$ whose union contains $[0, s-\varepsilon]$. Then $\mathcal{F}' \cup \{U\}$ has union containing $[0, s+\varepsilon]$. But then s is not the least upper bound of X. Hence $s = 1$.

Let $U \in \mathcal{F}$ be chosen with $s = 1 \in U$. Choose $\varepsilon > 0$ with the open set

$$(1-\varepsilon, 1] = [0,1] \cap (1-\varepsilon, 1+\varepsilon)$$

contained in $\mathcal{U}$. Let* $\mathcal{F}'$ be a finite subfamily of $\mathcal{F}$ whose union contains $[0, 1-\frac{\varepsilon}{2}]$. Then the union of the finite family $\mathcal{F}' \cup \{\mathcal{U}\}$ contains $[0,1]$. So $s = 1 \in X$.

*we can do this since $1-\frac{\varepsilon}{2} < s$.

$\boxed{\mathcal{M}}$