Problem A cinema determines that the average time a customer queues for a ticket is 10 mins, and the average wait for popcorn is 5 mins. The waiting times are independent, and modelled by an exponential probability distribution. What is the probability that a movie goer waits less than 20 minutes?

Sol^n

X = ticket wait

$$f_1(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{10} e^{-\frac{x}{10}} & \text{if } x \geq 0 \end{cases}$$

Y = popcorn wait

$$f_1(y) = \begin{cases} 0 & \text{if } y < 0 \\ \frac{1}{5} e^{-\frac{y}{5}} & \text{if } y \geq 0 \end{cases}$$

Joint probability density function

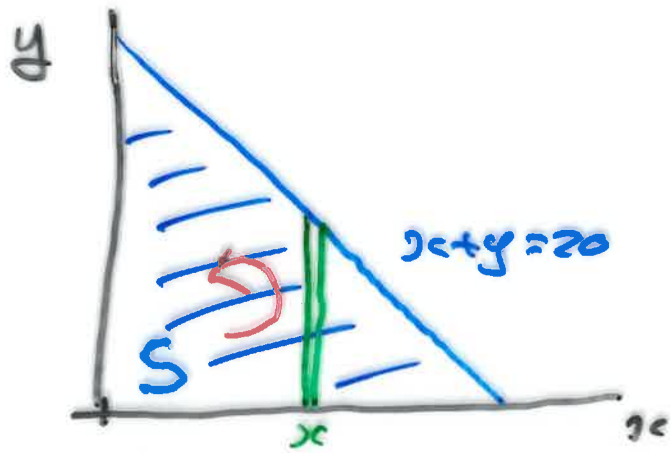
$$f(x,y) = f_1(x) f_2(y) = \begin{cases} \frac{1}{50} e^{-\frac{x}{10}} e^{-\frac{y}{5}} & x,y \geq 0 \\ 0 & \text{other} \end{cases}$$

$$\text{Prob}(X+Y < 20,\ X \geq 0,\ Y \geq 0) =$$

$$= I = \int_S \frac{1}{50} e^{-\frac{x}{10}} e^{-\frac{y}{5}} \, dx \wedge dy$$

where $S = \left\{ (x,y,z) \in \mathbb{R}^3 : \begin{array}{l} z = 0 \\ x,y \geq 0 \\ x+y < 20 \end{array} \right\}$

with positive orientation.



$$I = \int_0^{20} \left(\int_0^{20-x} \frac{1}{50} e^{-\frac{x}{10}} e^{-\frac{y}{5}} \, dy \right) dx$$

$$= \int_0^{20} e^{-\frac{x}{10}} (-5) e^{-\frac{y}{5}} \Big|_0^{20-x} dx$$

$$= \cdots$$

$$= 1 + e^{-4} - 2e^{-2}$$

$$\simeq 0.75$$

Differentiation of 1-forms

For a 1-form ω, and a 2-dimensional oriented region, we'd like to define a 2-form

$$d\omega$$

such

$$\boxed{\int_{\partial S} \omega = \int_S d\omega \qquad (*)}$$

The definition of $d\omega$ is determined by the desire to have equation $(*)$ hold.

Rules for differentiating 1-forms

For 1-forms ω and ω' and 0-forms $A, B, C, \ldots$ in variables $x, y, z \ldots$

1. $d(\omega + \omega') = d\omega + d\omega'$

2. $dA = \frac{\partial A}{\partial x} dx + \frac{\partial A}{\partial y} dy + \frac{\partial A}{\partial z} dz + \cdots$

3. $d(A\,dx + B\,dy + \cdots)$
$= (dA) \wedge dx + (dB) \wedge dy + \cdots$

4. $dx \wedge dx = 0, \; dy \wedge dy = 0, \ldots$

5. $dx \wedge dy = - dy \wedge dx$

6. $(\omega + \omega') \wedge dx = \omega \wedge dx + \omega' \wedge dx$

Example: Calculate $d\omega$ for

$$\omega = xy\,dz + yz\,dx + zx\,dy.$$

Sol^n

$$d\omega = d(xy\,dz) + d(yz\,dx) + d(zx\,dy)$$

$$= d(xy)\wedge dz + d(yz)\wedge dx + d(zx)\wedge dy$$

$$= (y\,dx + x\,dy)\wedge dz$$
$$+ (z\,dy + y\,dz)\wedge dx$$
$$+ (z\,dx + x\,dz)\wedge dy$$

$$= y\,dx\wedge dz + x\,dy\wedge dz$$
$$+ z\,dy\wedge dx + y\,dz\wedge dx$$
$$+ z\,dx\wedge dy + x\,dz\wedge dy$$

$$= -y\,\cancel{dz\wedge dx} + x\,\cancel{dy\wedge dz}$$
$$- z\,\cancel{dx\wedge dy} + y\,\cancel{dz\wedge dx}$$
$$+ z\,\cancel{dx\wedge dy} - x\,\cancel{dy\wedge dz}$$

$$= 0$$

<u>Exercise</u>: Find $d\omega$

where

$$\omega = A\,dx + B\,dy$$

with A, B functions of x and y.