Another model for projective geometry

1. Any two points are contained in a unique line.
2. Any two lines contain a unique point in common.
3. There exists at least one set of four points such that no three of them are on a line.
4. Every line contains at least three points.

The extended Euclidean plane

The Euclidean plane almost satisfies the axioms for projective geometry, with one exception. Parallel lines in the Euclidean plane never meet, and so there are pairs of lines with no point in common.

Suppose we attach an additional point to each Euclidean line and call this point the *point at infinity*. We attach this additional point to each Euclidean line so that all lines including parallel lines will intersect in one unique point. So the point that we attach to parallel lines must be the same.

For this reason it makes sense that the point which we attach is related to the slope of the line - slope is one feature that parallel lines share. So for all lines except vertical lines, we attach a ‘point at infinity’ that is their slope. Vertical lines have undefined slope so we just attach some symbol to indicate that the line is vertical.

Finally, we say all of this ‘points at infinity’ are said to be on the same line, called the *line at infinity*.

As an exercise, verify that this is a model for projective geometry.

Projective transformations of a projective line

Return to the real projective plane. Recall that the "point at infinity" depends on the viewing plane, and become visible by, for example, rotating the viewing plane. Let's consider different views of the projective line.

The mapping that takes us from one view to another is called a **Projective Transformation**.

There are **three** basic types of projective transformation.

1. Scaling
2. Inversion
3. Translation

1. Scaling

This happens when we "pull the viewing plan back", or "push it out", keeping it parallel to the original plane.

$$x \to ax$$

for some scalar a.

2. Inversion

This happens when we change the angle of the viewing plane so that we can now see what was the 'point at infinity'.

The simplest case of this is

$$x \to \frac{1}{x}.$$

This mapping between the projective lines L_1 and L_2 sends 0 on one line to ∞ on the other.

3. Translation

We may want to shift our coordinates to the left or right to match up the origin with a particular point.

Think of this as sliding the viewing plane to one side without changing it.

$$x \to x + b.$$

What is the most general mapping got by combining these?

Let's try

$$f(x) = \frac{1}{x}$$
$$g(x) = x + 2$$
$$h(x) = 3x$$

What is the most general formula we could get if we combine (compose) these functions in any order, using each of them as many times as we like?

A function of the form
$$f(x) = \frac{ax+b}{cx+d}$$
is called a **Projective Transformation**.

A projective transformation projects one line onto another. Two points to note:

1. When working with projective transformations, we take $\frac{1}{0}$ to mean ∞, the point at infinity.
2. We have to add the condition $ad - bc \neq 0$ in order to be able to invert a projective transformation.

If $ad - bc = 0$, then every point is mapped to the same point (the function is constant.)

Inverse of a projective transformation

To find the inverse of a projective transformation

$$f(x) = \frac{ax + b}{cx + d}$$

we let $y = \frac{ax+b}{cx+d}$, and solve for x.

We get

$$x = \frac{-dy + b}{cy - a} \quad \text{i.e.,} \quad f^{-1}(y) = \frac{-dy + b}{cy - a}.$$

The inverse of a projective transformation is a projective transformation.