3. Projective Geometry

The Axioms for a Projective Plane

The undefined terms are **point** and **line**. The axioms are:

1. Any two points are contained in a unique line.
2. Any two lines contain a unique point in common.
3. There exists at least one set of four points such that no three of them are on a line.
4. Every line contains at least three points.

Motivation

Let's start with the idea of the all-seeing eye that can visualize a projection onto a plane. Think of an artist holding up a pane of glass in front of a scene and tracing the outlines of what she sees on the glass.

Imagine a beam of light coming from a point in the scene to the artist's eye, passing through a point on the glass.

The glass pane can be held in many different positions, giving lots of different projections of the same point. What they all have in common is the line (beam of light) from the scene to the eye.

Each projection is a different image of the same thing.

So we start in three-dimensional space $\mathbb{R}^3$ and we place the 'all-seeing eye' at the origin $\mathsf{O} = (0, 0, 0)$.

- A **Point** is, by definition, a line through O.
- A **Line** is, by definition, a plane through O in $\mathbb{R}^3$.

This gives us the model that has the features we are looking for. It is known as the Real Projective Plane and is denoted by $\mathbb{RP}^2$.

Visualizing the Real Projective Plane

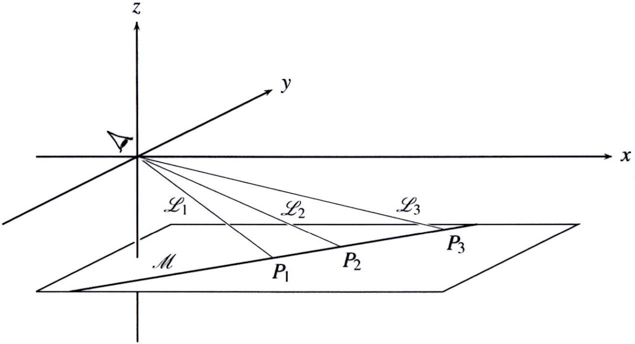
We have a description of the projective plane in terms of lines through the origin in three dimensional space.

It is difficult to "see" the entire projective plane as a "plane" all at once.

In fact, it is impossible to "embed" the projective plane faithfully into the conventional three-dimensional space.

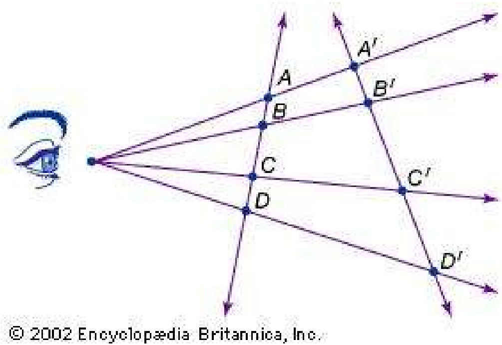
But we can look at projections onto planes in $\mathbb{R}^3$. In this way, we get "partial views" of the real projective plane.

Imagine that there is an "all seeing eye" at the origin of $\mathbb{R}^3$.

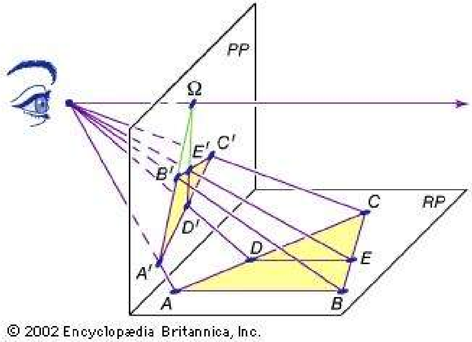


Points P_1, P_2, P_3 in the plane $z = -1$ are connected to the origin by lines $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$.

Different viewpoints



Angles are not preserved



The Horizon

In any projection, there is always one line we cannot see. It comes from the plane in $\mathbb{R}^3$ through the origin that is parallel to our viewing plane. This line is the **horizon**.

There is nothing special about this line. The horizon line changes from one viewpoint to another.

There are **no parallel lines**. Lines that appear parallel in one projection are said to "meet on the horizon".

Coordinate Geometry in the Real Projective Plane

We need coordinates in the Real Projective Plane $\mathbb{RP}^2$.

If (x, y, z) and (x', y', z') are the coordinates of two points on the same line through O and neither of them is $(0, 0, 0)$, then each of them must be a multiple of the other:

$$(x, y, z) = c(x', y', z') and (x', y', z') = c^{-1}(x, y, z)$$

for some $c \neq 0$.

We could use either of these as the coordinates of the projective point determined by this line. So (x, y, z) and (x', y', z') are considered to be equivalent, since they represent the same projective point.

These are called the **homogeneous** coordinates of the projective point.

Homogeneous coordinates

We use square brackets to signal that we are using homogeneous coordinates:

$$[x, y, z]$$

With this notation, $[x, y, z]$ is the same as $[cx, cy, cz]$ for $c \neq 0$.

For example, the points $(1, 2, 4)$, $(4, 8, 16)$ and $(-3, -6, -12)$ are all on the same line through O, so they all represent the same projective point in the Real Projective Plane.

We could denote this point by $(1, 2, 4)$, or $(4, 8, 16)$, or $(-3, -6, -12)$: they are all the same projective point.

$$[1, 2, 4] = [4, 8, 16] = [-3, -6, -12].$$

The Equation of a Projective Line

A **projective line** in $\mathbb{RP}^2$ is the same as a plane through O in $\mathbb{R}^3$:

$$ax + by + cz = 0.$$

This equation describes the plane in $\mathbb{R}^3$ through O that is perpendicular to the vector (a, b, c) (this vector is called a **normal** to the plane).

So, if we interpret x, y, z as homogeneous coordinates, this becomes the equation of a projective line in the Real Projective Plane.

Example: Consider the projective line given by

$$x - 2y + 5z = 0$$

- Find the coordinates of two points on this line.
- Sketch the projection of this projective line onto the plane parallel to, and one unit below, the xy-plane.