The Poincaré Disc Model for Hyperbolic Geometry

- The **Plane** is the region inside the unit circle in the plane $\mathbb{R}^2$. The points on the circumference of the circle are not included.
- The term **Point** has its usual meaning.
- The **(Straight) Lines** are:
  - The diameters of the circle;
  - Arcs of circles that intersect the boundary circle at right angles.

Some Theorems in Hyperbolic Geometry

We have to be careful not to use results from Euclidean geometry that rely on the Parallel Postulate. Here are some familiar examples from Euclidean geometry.

If a straight line cuts two parallel lines, then the alternate angles are equal to one another.

This is not true in hyperbolic geometry. Here is another example. The proof uses the result above.

The sum of the interior angles in a triangle equals $180°$.

We will use the following theorem from hyperbolic geometry. We won't study the proof.

Theorem (Hyperbolic Theorem 1 (No proof))

In the hyperbolic plane, the sum of the interior angles in a triangle is always less than $180°$.

We can now prove some interesting results.

Theorem (Hyperbolic Theorem 2)

In the hyperbolic plane, rectangles do not exist.

Proof:

A quadrilateral is said to be convex if the lines joining the corners are inside the quadrilateral.

Theorem (Hyperbolic Theorem 3)

In the hyperbolic plane, the sum of the interior angles in a convex quadrilateral is always less than 360°.

Proof:

Recall that two triangles are said to be **similar** if they have the same interior angles.

The next result is very surprising and completely inconsistent with our Euclidean spatial intuition.

Theorem (Hyperbolic Theorem 4)

In the hyperbolic plane, if two triangles are similar, then they are congruent.

Proof:

This tells us that it is impossible to magnify or shrink a triangle without distortion.

http://pi.math.cornell.edu/~mec/Winter2009/Mihai/index.html