2. Hyperbolic Geometry

Hyperbolic Geometry

An axiom is said to be **Independent** if it is not possible to prove it from the other axioms.

To show that an axiom is independent: Find a model for the other axioms in which this axiom is not true.

Euclid's Fifth Axiom (the Parallel Postulate) is the classic example of an axiom that is independent of the others.

This was shown conclusively in the 19th century, when Gauss, Bolyai and Lobachevsky developed 'non-euclidean geometries' in which the Parallel Postulate is not assumed as an axiom.

The Hyperbolic Plane - An Example of Non-Euclidean Geometry

In Hyperbolic Geometry, the Parallel Postulate is not assumed. It is replaced by the following axiom:

If ℓ is a line and P is a point that is not on ℓ, then there are infinitely many lines through P that are parallel to ℓ.

(Here, by parallel we mean: the lines never meet.)

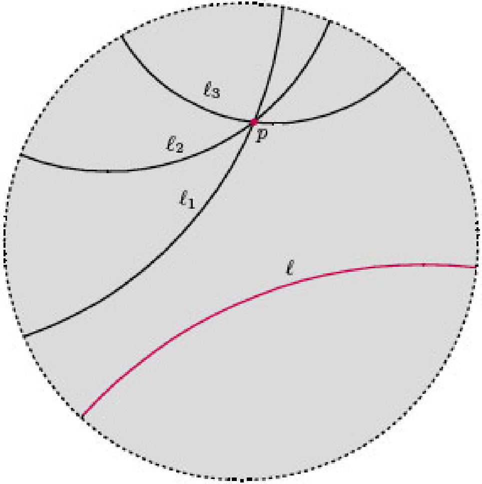
This is very hard for us to imagine!

The Poincaré Disc Model for Hyperbolic Geometry

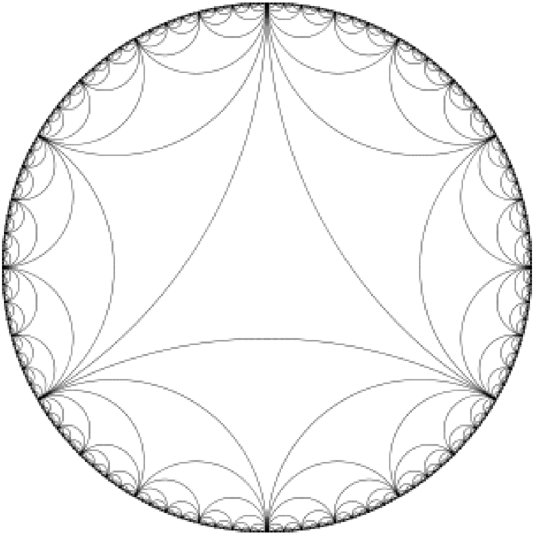
- The **Plane** is the region inside the unit circle in the plane R2. The points on the circumference of the circle are not included.
- The term **Point** has its usual meaning.
- The **(Straight) Lines** are:
  - The diameters of the circle;
  - Arcs of circles that intersect the boundary circle at right angles.

Note: The angle between two curves that intersect at a point is defined to be the angle between their tangent lines at that point.

Parallel lines in the Hyperbolic Plane



Triangles in the Hyperbolic Plane



Distance in the Hyperbolic Plane

Let P, Q be two points in the Hyperbolic Plane.

In Euclidean geometry, the distance between these points is the length of the straight line that joins them. We'll denote the Euclidean distance by $|\mathsf{PQ}|$.

In the Hyperbolic Plane, we use a completely different measure of distance.

Take the Euclidean line through the points P, Q, i.e., the chord of the circle containing these points.

Let A, B be the points where it meets the boundary circle.

Label these points so A, P, Q, B appear in that order.

The Hyperbolic Distance between P and Q is defined by

$$d(P, Q) = \log \frac{|AQ||PB|}{|AP||QB|}$$

where log (or ln) is the natural logarithmic function.

Examples

Calculate the hyperbolic distance between P and Q where

1. $P = (0, 0)$, $Q = (0.1, 0)$.
2. $P = (0.1, 0)$, $Q = (0.2, 0)$.
3. $P = (0.4, 0)$, $Q = (0.5, 0)$.
4. $P = (0.8, 0)$, $Q = (0.9, 0)$.

Next, let's calculate the hyperbolic distance from the centre of the boundary circle to an arbitrary point.

Take $P = O$ to be the centre and let Q be on the circle with radius r. So we have $0 \leqslant r < 1$. We get

$$d(O, Q) = \log(\frac{1+r}{1-r})$$

Consider what happens as $r \to 1$. Then $d(O, Q) \to \infty$.

Conclusion: the boundary is infinitely far from the centre. This is not apparent, as we are seeing at a 'distorted' view of the hyperbolic plane.