

The axioms of Group Theory

A **Group** is a set, G , with a ‘composition’ operation that takes every pair of elements x, y of G to an element $x \circ y$ of G , with the following properties:

- 1 There is an element e of G (called the **identity element**), such that $x \circ e = e \circ x = x$ for every x in G .
- 2 (**Associativity.**) For all x, y, z in G , we have $x \circ (y \circ z) = (x \circ y) \circ z$.
- 3 For every x in G , there is an element y of G such that $x \circ y = y \circ x = e$ (y is the **inverse** of x and is often denoted by x^{-1} or $-x$.)

Some examples of groups

- The Integers, with addition as the operation.
- The non-zero Real Numbers with multiplication.
- The 2×2 invertible matrices with the usual matrix multiplication.
- The six symmetries of an equilateral triangle, where the composition of two symmetries is given by performing one before the other.

The group of symmetries of an equilateral triangle

A **symmetry** is a transformation (or mapping) that preserves the shape of the triangle. There are six in all:

- The identity transformation e (nothing moves).
- Three reflectional symmetries, X , Y , Z (reflections in the lines through one point and the mid-point of the line joining the other two points).
- Two rotational symmetries R , R^2 (Rotating by 120° once or twice anticlockwise around the center point of the triangle).

Verify this is a group under composition of symmetries, and write the **group multiplication table**. 

Models

A **Model** for a set of axioms is a set, along with an interpretation of the undefined terms, in which all the axioms are true. A model can be thought of as an ‘example’ of a situation in which all the axioms hold.

Examples of models:

- \mathbb{R}^2 , the xy -plane with the normal interpretations of points and lines, is a model for the axioms of Euclidean geometry.
- The set of natural numbers as we commonly understand it, $\{0, 1, 2, 3, \dots\}$ is a model for the Peano Axioms. We interpret $S(n)$ as $n + 1$.
- We have seen four different models for the axioms of group theory.

An axiomatic system is called **Categorical** if it has a unique model.

‘Unique’ means that any two models are **isomorphic**: the two models are identical, except for the names of the elements.

For example, the axioms for Group Theory are **not** categorical: we have seen that there are at least four different models.

Are the Peano Axioms categorical? Strangely, the answer is no! There are ‘**Non-Standard Models**’ of the natural numbers. They are very difficult to describe or to picture.

Euclidean Geometry, as described by Hilbert’s axioms, is categorical. (**Exercise:** Have a look at the wikipedia page for Hilbert’s axioms, or any other source in the library.)

Completeness and Consistency

In Mathematics, when we have a body of knowledge about something, we often try to capture this knowledge in a set of axioms.

Doing this puts our knowledge on a solid footing. We use the axioms to prove theorems. These theorems are statements that will be true in every model of the system.

Ideally, we would like our axiom system to have two important properties: Completeness and Consistency.

Completeness:

If a statement is true in every model of the system, then it can be proved using the axioms. This is a desirable property, but in many cases, we don't know if it holds.

Consistency:

This is the requirement that the axioms do not contain any hidden contradictions.

Here are some equivalent ways to say that a system of axioms is consistent:

- It is not possible to prove both a statement and its negation.
- It is not possible to prove the negation of one of the axioms using the other axioms.
- The system has a model.

We have models for Euclidean geometry, the Peano Axioms and Group Theory. So we know that these systems are consistent.

An Example of an Inconsistent System of Axioms

The undefined terms are ‘point’ and ‘special set’.

The axioms:

- 1 Every special set contains exactly three points.
- 2 There are exactly four points.
- 3 Every point is contained in some special set.
- 4 No point is contained in more than one special set.

Why is this inconsistent? 

Independence

An axiom is said to be **Independent** if it is not possible to prove it from the other axioms.

To show that an axiom is independent: Find a model for the other axioms in which this axiom is not true.

Euclid's Fifth Axiom (the Parallel Postulate) is the classic example of an axiom that is independent of the others.

This was shown conclusively in the 19th century, when Gauss, Bolyai and Lobachevsky developed 'non-euclidean geometries' in which the Parallel Postulate is not assumed as an axiom. (We will discuss hyperbolic geometry next!)