

MA334 Geometry

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1. Axiom systems

Axiom systems

- **Undefined terms:** These are words or phrases that are given without definition. The only properties they have that we may use are those given in the axioms.
- **Axioms (postulates):** Statements that we accept without proof.
- **Theorems:** Statements that can be proved using only the given axioms and logical deduction.

Example

A committee structure

Undefined terms: Committee. Member.

Axioms:

- 1 Each committee is a set with three members.
- 2 Each member belongs to exactly two committees.
- 3 No two members may be together on more than one committee.
- 4 There is at least one committee.

Models

An axiomatic system is not very interesting if it describes something that never happens.

A **Model** for an axiomatic system is a context in which we provide an interpretation of the undefined terms in such a way that all the axioms say things that are true.

Let's look again at the 'Committee Structure' axiomatic system. Why must there be more than five members? 📝

A model for the Committee Structure axiom system

In this model there are six Members: A, B, C, D, E, F.

A Committee is a subset of the set of Members.

Example

Consider the four committees

- 1 {A, B, C},
- 2 {A, D, E},
- 3 {B, D, F},
- 4 {C, E, F}.

Verify this satisfies the axioms 

Exercise: Find another model.

Another example

The Peano Axioms for Arithmetic

In 1889, Giuseppe Peano published a set of axioms that describe the set \mathbb{N} of natural numbers. His axioms are enough to prove all the standard facts about arithmetic for natural numbers. Note: there are two common definitions of the set of natural numbers:

$$\{0, 1, 2, \dots\} \quad \text{and} \quad \{1, 2, 3, \dots\}$$

We'll use the first one.

The most fundamental property of the natural numbers is the ordering: every natural number n is followed by its 'successor', $n + 1$. Peano denoted this by $S(n)$.

The undefined terms are natural number, 0 and S.

The Peano Axioms:

- 1 0 is a natural number.
- 2 If n is a natural number, then $S(n)$ is a natural number.
- 3 If m, n are natural numbers then $S(m) = S(n)$ if and only if $m = n$.
- 4 There is no natural number n satisfying $S(n) = 0$.
- 5 Let P be a set of natural numbers such that: 0 is in P , and if n is in P , then $S(n)$ is in P . Then P contains every natural number. (The Principle of Induction.)

Note: Peano had four other axioms, which are now usually omitted as unnecessary

- ⑥ For every natural number n , we have $n = n$. (Equality is **reflexive**.)
- ⑦ For all natural numbers m and n , if $m = n$, then $n = m$. (Equality is **symmetric**.)
- ⑧ For all natural numbers m , n , p , if $m = n$ and $n = p$, then $m = p$. (Equality is **transitive**.)
- ⑨ For all m , n , if m is a natural number and $m = n$, then n is a natural number.

All the standard operations and procedures can now be defined using these axioms. We define

$$1 = S(0), 2 = S(1) = S(S(0)) \text{ etc.}$$

Addition

We define addition recursively with the following definitions:

- ① $n + 0 = n$.
- ② $n + S(m) = S(n + m)$.

So, for example, we get

$$n + 1 = n + S(0) = S(n + 0) = S(n).$$

and

$$n + 2 = n + S(1) = S(n + 1) = S(S(n))$$

and so on. Note how each step in these calculations can be justified by appealing to the definition and the axioms.

Multiplication

We make the following definition:

- ① $n \cdot 0 = 0$.
- ② $n \cdot S(m) = n + n \cdot m$.

Inequality

We define $m \leq n$ if and only if there is a natural number p such that $m + p = n$.

All the usual properties can now be proved. Some examples:

- ① Addition and multiplication are commutative: $m + n = n + m$ and $m \cdot n = n \cdot m$.
- ② Additive and multiplicative identity elements: $n + 0 = n$ and $m \cdot 1 = m$.
- ③ If $m \leq n$ then $m + p \leq n + p$ and $m \cdot p \leq n \cdot p$.

The axioms of Group Theory

A **Group** is a set, G , with a ‘composition’ operation that takes every pair of elements x, y of G to an element $x \circ y$ of G , with the following properties:

- 1 There is an element e of G (called the **identity element**), such that $x \circ e = e \circ x = x$ for every x in G .
- 2 (**Associativity.**) For all x, y, z in G , we have $x \circ (y \circ z) = (x \circ y) \circ z$.
- 3 For every x in G , there is an element y of G such that $x \circ y = y \circ x = e$ (y is the **inverse** of x and is often denoted by x^{-1} or $-x$.)

Some examples of groups

- The Integers, with addition as the operation.
- The non-zero Real Numbers with multiplication.
- The 2×2 invertible matrices with the usual matrix multiplication.
- The six symmetries of an equilateral triangle, where the composition of two symmetries is given by performing one before the other.