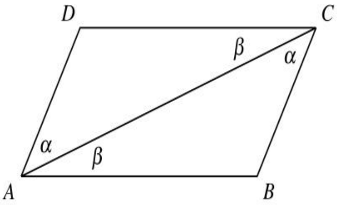
Euclid's understanding of area

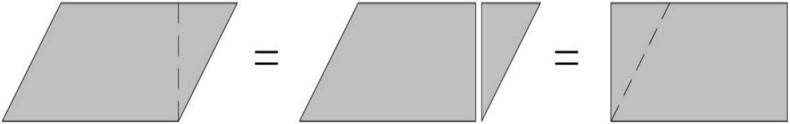


The triangles are congruent.

So they have the same area.

Therefore the area of the triangle is half the area of the parallelogram.

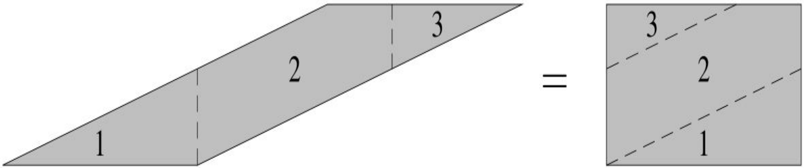
A parallelogram has the same area as a rectangle on the same base:



Cut up one area and reassemble it to make the other.

Can you see the flaw in this argument?

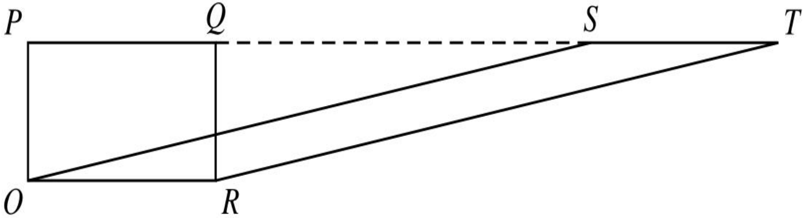
It doesn't work in this case



A more complicating cutting and assembling is needed.

We night need even more than two cuts.

If we can both add and subtract areas, then one construction suffices:



We add and subtract congruent triangles.

Putting these facts together:

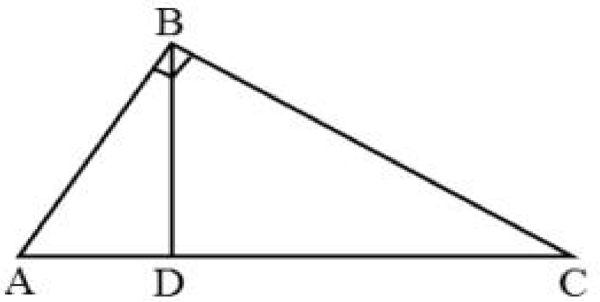
A triangle has half the area of any parallelogram standing on the same base and contained between the same parallels.

"Things that are equal to the same thing are equal to one another" – one of the "Common Notions".

In particular, we get:

A triangle has half the area of the rectangle with the same base and the same height.

Pythagoras' Theorem: the Project Maths proof

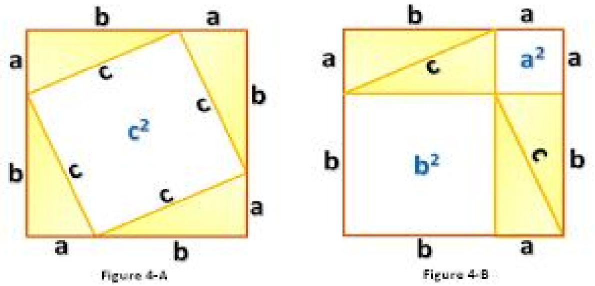


The proof starts with the observation that the triangles

$$\Delta ABC,\ \Delta ADB,\ \text{and}\ \Delta BDC$$

are **similar**.

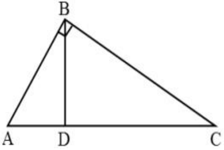
Pictorial proof of Pythagoras' Theorem



Euclid's Second Proof of Pythagoras (Book 6, Proposition 31)

Let ΔABC be a right angled triangle with hypotenuse AC.

Draw the perpendicular BD from B to AC.



The triangles ΔABC, ΔADB and ΔBDC are similar. And we have

$$|\Delta ABC| = |\Delta ADB| + |\Delta BDC|$$

where $|\Delta ABC|$ denotes the area of the triangle ΔABC.

We now look at how the areas of the squares on the hypotenuses of these similar triangles are related to the areas of the triangles themselves.

The key point:
As the triangles are similar, the ratio between the area of the square on the hypotenuse and the area of the triangle is the same in each case.

Numbers in Euclid's Elements

Numbers are studied in Books VII to IX of Euclid's Elements. This is what we would now call **Arithmetic** or **Number Theory**. There is nothing of what we would call Algebra: no use of symbols to represent a variable, no manipulation of equations, etc.

This makes these books very hard for a modern reader to follow. Euclid's numbers are based on a "unit" (1), which can be duplicated (2), triplicated (3), subdivided by bisection (1/2), trisection (1/3) etc.

We would say that Euclid knew the Natural Numbers and the Rational Numbers.

There is no zero number in the Elements.

Prime Numbers

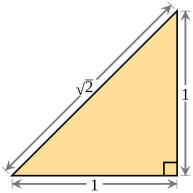
"A prime number is that which is measured by a unit alone"

Book IX, Proposition 20 states that there are infinitely many prime numbers. ("Prime numbers are greater than any assigned multitude of prime numbers").

Proof:
Let $p_1, \ldots, p_n$ be n different prime numbers and consider the number $p_1 p_2 \cdots p_n + 1$.

- If this number is prime, then there are at least $n + 1$ primes.
- If it is not prime, then it is divisible by a prime number. It is not divisible by any of the primes $p_1, \ldots, p_n$, since each of these leaves a remainder of 1 when divided into this number. So there exists another prime number.

Irrational numbers (Incommensurables)



It is easy to see that $\sqrt{2}$ is constructible. However, $\sqrt{2}$ is **irrational**, or **incommensurable** in Euclid's language.

The discovery of this fact dates back to the 5th century BC, when it was known to the school of Pythagoras. (Euclid's Elements are around 300 BC)

It is not mentioned explicitly in the Elements.

The Pythagorean Proof of the Irrationality of $\sqrt{2}$: