Theorems in Euclid's elements

https://mathcs.clarku.edu/~djoyce/elements/bookI/bookI.html

Propostion 4 (Theorem):
If two triangles ABC, DEF have two sides AB, AC respectively on one equal to two sides DE, DF of the other, and have also the angles A and D included by those sides equal, then the triangles are equal in every respect. Their bases or third sides BC, EF are equal and the angles B, C at the base are respectively equal to the angles E, F at the base of the other; namely, those shall be equal to which the opposite sides are equal.

Proof: If the triangle ABC is **superposed** on the triangle DEF, and if the point A is placed on the point D and the straight line AB on DE, then the point B also coincides with E, because AB equals DE....

...(cont)... Again, AB coinciding with DE, the straight line AC also coincides with DF, because the angle BAC equals the angle EDF. Hence the point C also coincides with the point F, because AC again equals DF.

But B also coincides with E, hence the base BC coincides with the base EF and equals it.

Thus the whole triangle ABC coincides with the whole triangle DEF and equals it.

And the remaining angles also coincide with the remaining angles and equal them, the angle ABC equals the angle DEF, and the angle ACB equals the angle DFE. □

Comments on this proof

Is it clear what we mean by "superposing" one triangle on another one?

We might picture this as "sliding" one of the triangles over in a continuous motion. But if one of the triangles is the mirror image of the other, then this is not physically possible.

One way around this is to add some axioms relating to a group of transformations acting on the plane (we would need reflections, rotations, glides,...)

Hilbert's solution was to make the theorem one of his axioms.

This is the first of three theorems about **congruence** of triangles. The others are Proposition 8 and Proposition 26.

Proposition 7:
If two triangles ACB, ADB are on the same base AB and on the same side of it have one pair of coterminous sides AC, AD equal to one another, then the other pair of coterminous sides BC, BD must be unequal.

Proof (Summary): ACD is an isosciles triangle and so the angles $\angle ADC$ and $\angle ACD$ are equal.

The angle $\angle$ADC is greater than $\angle$BDC. Therefore the angle $\angle ACD$ is greater than $\angle$BDC. So the angle $\angle$BCD is even greater than the angle $\angle$BDC.

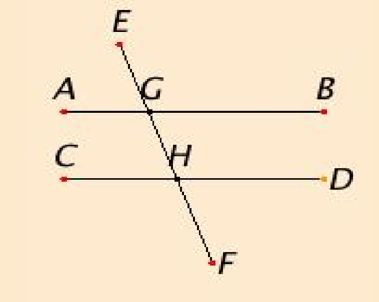
If BD were equal to BC, then the angle $\angle$BCD would be equal to $\angle$BDC. But this is not the case. Therefore BD is not equal to BC.

Proposition 8:
If two triangles ABC and DEF have two sides AB, AC of one respectively equal to two sides DE, DF of the other, and have also the base BC of one equal to the base EF of the other, then the two triangles are equal, and the angles of one are respectively equal to the angles of the other – namely, those are equal to which the equal sides are opposite.

This follows immediately from Proposition 7.

Proposition 29 (Theorem):

A straight line falling on parallel straight lines makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the sum of the interior angles on the same side equal to two right angles.



This is the first proof that uses the Parallel Axiom.

Euclid's statement of the Parallel Axiom

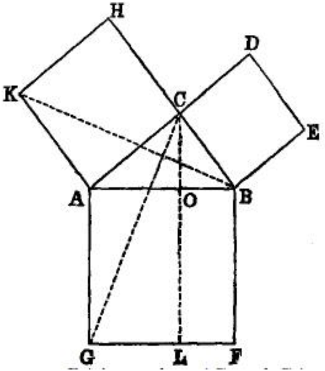
If two right lines meet a third line so as to make the sum of the two interior angles on the same side less than two right angles, these lines being produced shall meet at some finite distance.

There are many equivalent ways to state this axiom. One of the most popular is:

Playfair's Axiom:
There is exactly one line parallel to a given line passing through a given point.

Pythagoras' Theorem

Proposition 47:
In a right angled triangle the square on the side opposite the right angle equals the sum of the squares on the sides containing the right angle.



Proof (from Casey's Elements):

1. *Construction: divide the square on the hypotenuse into two rectangles. (The plan is to show that each of the rectangles has the same area as one of the smaller squares.)*

On the sides AB, BC, CA describe squares [Proposition 46]. Draw CL parallel to AG. Join CG, BK.

2. *Verify that the diagram is correct.*

Then because the angle $\angle ACB$ is right (hyp.), and $\angle ACH$ is right, being the angle of a square, the sum of the angles $\angle ACB$ and $\angle ACH$ is two right angles; therefore BC, CH are in the same right line [Proposition 14]. In like manner AC, CD are in the same right line. Again, because $\angle BAG$ is the angle of a square it is a right angle: in like manner $\angle CAK$ is a right angle.

3. *Identify a pair of congruent triangles.*

Hence $\angle BAG$ is equal to $\angle CAK$: to each add $\angle BAC$, and we get the angle $\angle CAG$ equal to $\angle KAB$. Again, since $\square BG$ and $\square CK$ are squares, BA is equal to AG, and CA to AK. Hence the two triangles CAG, KAB have the sides CA, AG in one respectively equal to the sides KA, AB in the other, and the contained angles $\angle CAG$, $\angle KAB$ also equal. Therefore [Proposition 4] the triangles are equal.

4. *Relate the areas of the two triangles to the relevant areas.*

But the parallelogram $\lozenge AL$ is double of the triangle CAG [Proposition 41], because they are on the same base AG, and between the same parallels AG and CL. In like manner the parallelogram $\lozenge AH$ is double of the triangle KAB, because they are on the same base AK, and between the same parallels AK and BH; and since doubles of equal things are equal, the parallelogram $\lozenge AL$ is equal to $\lozenge AH$.

5. *Repeat.*

In like manner it can be proved that the parallelogram $\Diamond$BL is equal to $\Diamond$BD. Hence the whole square $\Box$AF is equal to the sum of the two squares $\Box$AH and $\Box$BD. $\Box$

Note:
Later in the Elements (Book 6, Proposition 31) Euclid gave a completely different proof of Pythagoras' Theorem that reveals more than is apparent from his first proof.

Book 6, Proposition 31:
In a right angled triangle, the figure on the side opposite the right angle equals the sum of the similar and similarly described figures on the other two sides.