Class test 2: Friday 29 Mar, AC203

- Problem sheet parts IV - V (up to Q4).
- Lectures 12-18.
- Short questions, may include theorems, proofs, examples, calculations.
- Bring paper to write on and a calculator.

Ruler and Compass Constructions

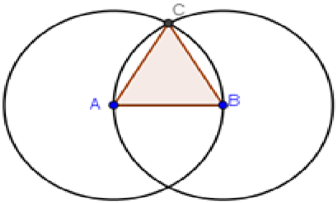
- A **ruler** (straight edge). Used to construct the line between two given points. Note: there are no markings on the ruler. But we can choose a "unit length" and replicate it – this is one of Euclid's constructions.
- A **compass**. Used to construct the circle, given the centre and radius.

Propositions in this lecture are numbered according to Joyce's online version of Elements.

https://mathcs.clarku.edu/~djoyce/elements/elements.html

We have already seen that Euclid's first "Proposition" was a construction.

Proposition 1:
On a given finite line AB to construct an equilateral triangle.



Proposition 9:
To bisect an angle given $\angle BAC$.

1. Using the compass, mark a point D on AB and a point E on AC such that $|AD| = |AE|$.
2. Construct a circle centred at D with radius $|DE|$, and a circle centred at E with radius $|DE|$. Mark the intersection points.
3. The line passing through the intersection points is the set of points equidistant from D and E, and also passes through A. This line bisects the angle $\angle BAC$.

Proposition 10:
To bisect a given line segment AB.

1. Repeat steps 2 and 3 from the previous slide.
2. The line of points equidistant from D and E bisects the line segment DE. The same applies to any line segment AB.

Proposition 11: Given a point C on a line AB, to construct a line through C perpendicular to AB.

Solution: With the compass, mark two points D and E on the line AB. Then bisect the line segment DE.

Proposition 12: Given a point C *not* on a line AB, to construct a line through C perpendicular to AB. (Exercise)

Construction regular n-gons

A regular n-gon is a convex figure bounded by n sides that are lines of equal length.

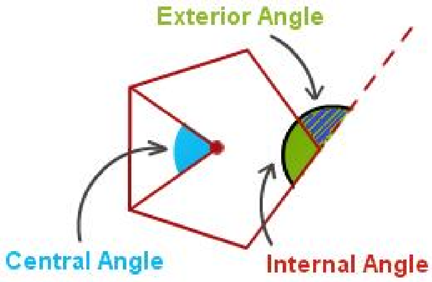
Proposition 46:
To construct a square on a given line AB. (Exercise)

How can we construct a regular pentagon (5 equal sides)? This is hard.

But it is easy to construct a regular hexagon (6 equal sides).

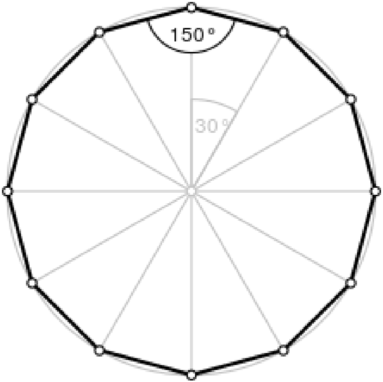
Some Regular Polygons are easy to construct

The angle at the centre of a regular n-gon subtended by each side is $360°/n$. If we can construct the central angle (or the internal/external angle), constructing the n-gon follows.



For the Hexagon, this angle is $60°$, the same as the angle in an equilateral triangle. So it is easy to construct a Hexagon.

For a regular $2n$-gon, the central angle is half the central angle of a regular n-gon.
We can bisect any angle. So, if the regular n-gon can be constructed, then so can the regular $2n$-gon.



Constructible Polygons

Euclidean geometers knew how to construct the Square, Equilateral Triangle and Regular Pentagon.

And they knew how to construct a regular polygon with double the number of sides of a given constructible regular polygon.

Is it possible to construct every regular polygon?

This problem was unsolved for over 2,000 years.

As of today, only 31 constructible regular polygons with an **odd** number of sides are known.

We want to be able to construct the central angle, $2\pi/n$, for a regular n-gon.

If we could construct a line of length $\cos(2\pi/n)$, then we could do this:

In 1796, at the age of 19, Carl Friedrich Gauss gave a construction for the regular 17-gon.

He went on to show that in general, the number $\cos(2\pi/n)$ is constructible if and only if it can be written in terms of the four basic operations of arithmetic and square roots.

The Gauss-Wantzel Theorem

Theorem

A regular n-gon is constructible if and only if

$$n = 2^k p_1 \cdots p_m$$

where k is a natural number and $p_1, \ldots, p_m$ are prime numbers of the form $p = 2^{(2^a)} + 1$.

Prime numbers of this form are called **Fermat Primes**.

Examples: 3, 5, 17, 257, 65537.

These are the only known Fermat Primes, as of today.