5. Euclidean Geometry

Topics

- History of Euclidean Geometry from Euclid to Hilbert.
- Some highlights, including Pythagoras' Theorem.
- How number/distance was understood in Greek mathematics – "commensurables" and "incommensurables".
- The Parallel Axiom.
- Euclidean Geometry in second level education in Ireland.

Some resources

Look for these sources on Blackboard or online.

- Ricardo Nirenberg's lecture on The Axiomatic Method.
- Wikipedia article on Euclid's Elements.
- Euclid's Elements — a classical textbook (Casey, 1885), in use up to relatively recently.
- An article on the many (over 100) proofs of Pythagoras' Theorem.

Casey's version of Elements

Since 1885, Casey's book has been in use until relatively recently. It begins with 34 definitions.

Examples:

- A *point* is that which has position but not dimensions.
- A *line* is length without breadth.
- A line which lies evenly between its extreme points is called a *straight* or *right line*.
- A *surface* is that which has length and breadth.
- The inclination of two right lines extending out from one point in different directions is called a *rectilineal angle*.

Postulates and axioms

Postulates and axioms are essentially assumptions or propositions that are not proven but are held to be self-evident. In Casey's book there are 3 postulates followed by 12 axioms, many of which are not specific to geometry.

Example

If equals be added to equals the sums will be equal.

The 5 axioms of Euclidean geometry which we use are contained in these postulates and axioms.

Propositions

Propositions are framed as problems with solutions and demonstrations, with references to the definitions, axioms and postulates .

PROP. I. - Problem: *On a given finite right line (AB) to construct an equilateral triangle.*

Sol. - With A as centre, and AB as radius, describe the circle BCD (Post. iii.). With B as centre, and BA as radius, describe the circle ACE, cutting the former circle in C. Join CA, CB (Post. i.). Then ABC is the equilateral triangle required.

Dem. - Because A is the centre of the circle BCD, AC is equal to AB (Def. xxxii.). Again, because B is the centre of the circle ACE, BC is equal to BA. Hence we have proved.

$$AC = AB$$

and

$$BC = AB.$$

But things which are equal to the same are equal to one another (Axiom i.); therefore AC is equal to BC; therefore the three lines AB, BC, CA are equal to one another. Hence the triangle ABC is equilateral (Def. xxi.); and it is described on the given line AB, which was required to be done.

A flaw

There is actually a flaw in this proof. Remember we assume only the axioms and postulates that are given.

How do we know the two circles intersect?

This, (and other issues), were eventually resolved by David Hilbert in the 20th Century by the addition of some additional axioms.

PROP. II. - Problem: *From a given point* A *to draw a right line equal to a given finite right line* BC.

Sol. - Join AB (Post. i.); on AB describe the equilateral triangle ABD [Prop. I.]. With B as centre, and BC as radius, describe the circle ECH (Post iii.). Produce DB to meet the circle ECH in E (Post. ii.). With D as centre, and DE as radius, describe the circle EFG (Post. iii.). Produce DA to meet this circle in F. AF is equal to BC.

Dem. - Because D is the centre of the circle EFG, DF is equal to DE (Def. xxxii.). And because DAB is an equilateral triangle, DA is equal to DB (Def. xxi.). Hence we have

$$\mathsf{DF} = \mathsf{DE}$$

and

$$\mathsf{D}A = \mathsf{DB}$$

and taking the latter from the former, the remainder AF is equal to the remainder BE (Axiom iii.). Again, because B is the centre of the circle ECH, BC is equal to BE; and we have proved that AF is equal to BE; and things which are equal to the same thing are equal to one another (Axiom i.). Hence AF is equal to BC. Therefore from the given point A the line AF has been drawn equal to BC.