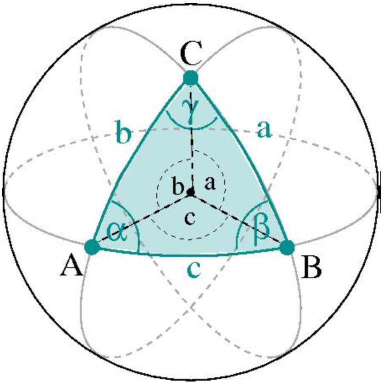
Spherical cosine rule



$$\cos(a) = \cos(\alpha)\sin(b)\sin(c) + \cos(b)\cos(c).$$

Here's another proof of the Spherical Cosine Rule that does not use the cross product:

Let's place the coordinate axes as follows: Let A be on the z axis. Thus

$$A = (0, 0, 1).$$

Now rotate the x and y axes so that B lies in the xz plane. Then the spherical coordinates of B are $\phi = c$ and $\theta = 0$. So the cartesian coordinates of B are

$$\mathrm{B} = (\cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi)) = (\sin(c), 0, \cos(c)).$$

Now the point C has spherical coordinates $\phi = b$ and $\theta = \alpha$. Thus

$$C = (\cos(\alpha)\sin(b), \sin(\alpha)\sin(b), \cos(b))$$

Now the angle between the vectors B and C is a. And these are unit vectors, so $B \cdot C = |B||C| \cos(a) = \cos(a)$. Therefore

$$\begin{aligned} \cos(a) = B \cdot C &= (\sin(c), 0, \cos(c)) \cdot (\cos(\alpha)\sin(b), \sin(\alpha)\sin(b), \cos(b)) \\ &= \cos(\alpha)\sin(b)\sin(c) + \cos(b)\cos(c). \end{aligned}$$

The Spherical Sine Rule

$$\frac{\sin(\alpha)}{\sin(a)} = \frac{\sin(\beta)}{\sin(b)} = \frac{\sin(\gamma)}{\sin(c)}$$

This is proved from the Spherical Cosine Rule:

$$\cos(a) = \cos(\alpha)\sin(b)\sin(c) + \cos(b)\cos(c)$$

gives

$$\cos(\alpha) = \frac{\cos(a) - \cos(b)\cos(c)}{\sin(b)\sin(c)}$$

and so

$$\sin^2(\alpha) = 1 - \cos^2(\alpha) = 1 - \left(\frac{\cos(a) - \cos(b)\cos(c)}{\sin(b)\sin(c)}\right)^2.$$

Therefore

$$\sin^2(\alpha) = \frac{\sin^2(b)\sin^2(c) - (\cos(a) - \cos(b)\cos(c))^2}{\sin^2(b)\sin^2(c)}.$$

So by expanding, $\frac{\sin^2(\alpha)}{\sin^2(a)}$ is equal to

$$\frac{\sin^2(b)\sin^2(c) - \cos^2(a) - \cos^2(b)\cos^2(c) + 2\cos(a)\cos(b)\cos(c)}{\sin^2(a)\sin^2(b)\sin^2(c)}.$$

Using $\sin^2(b) = 1 - \cos^2(b)$ and $\sin^2(c) = 1 - \cos^2(c)$ this is equal to

$$\frac{1 - \cos^2(a) - \cos^2(b) - \cos^2(c) + 2\cos(a)\cos(b)\cos(c)}{\sin^2(a)\sin^2(b)\sin^2(c)}.$$

So $\frac{\sin^2(\alpha)}{\sin^2(a)}$ may be expressed as

$$\frac{1 - \cos^2(a) - \cos^2(b) - \cos^2(c) + 2\cos(a)\cos(b)\cos(c)}{\sin^2(a)\sin^2(b)\sin^2(c)}.$$

Note the symmetry in a, b, c: $\frac{\sin^2(\beta)}{\sin^2(b)}$ and $\frac{\sin^2(\gamma)}{\sin^2(c)}$ lead to exactly the same expression. Therefore

$$\frac{\sin^2(\alpha)}{\sin^2(a)} = \frac{\sin^2(\beta)}{\sin^2(b)} = \frac{\sin^2(\gamma)}{\sin^2(c)}$$

Take square roots to get the Spherical Sine Rule.

Example

Let's use the Spherical Cosine Rule to find the shortest distance between Chicago $41.5°$N, $87.45°$W and Washington $38.55°$N, $77°$W.

Let ABC be the spherical triangle that has the North Pole at A, with B and C corresponding to Washington and Chicago respectively. We'll work on the unit circle and multiply the result by the Earth's radius, 6377.5 km.

We are looking for the distance a.
We need α, b and c to use the Spherical Cosine Rule. α is the difference between the longitudes of B and C. Thus

$$\alpha = 87.45° - 77° = 10.45°.$$

Since A is the North Pole, b and c are given by the spherical coordinate ϕ. Therefore

$$b = 90^\circ - 38.55^\circ = 51.45^\circ$$
$$c = 90^\circ - 41.5^\circ = 48.5^\circ.$$

So

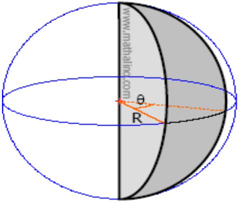
$$\cos(a) = \cos(10.45^\circ)\sin(51.45^\circ)\sin(48.5^\circ) + \cos(51.45^\circ)\cos(48.5^\circ)$$

This gives $\cos(a) = 0.9889\ldots$ and taking the inverse cosine, we get $a = 0.1487\ldots$ in radians.

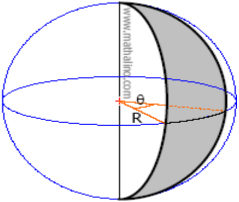
Multiplying by the radius of the Earth, 6377.5 km, we find that the distance is 948.54 km.

The Area of a Spherical Triangle

We start with the area of a *Lune*, the region between two great circles:



Spherical Wedge



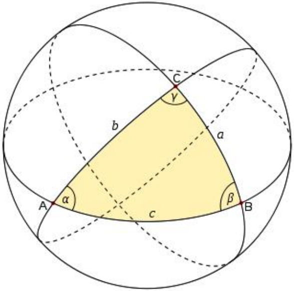
Spherical Lune

The total surface area of a sphere is $4\pi r^2$. So the area of a lune is

$$4\pi r^2 \times \frac{\theta}{2\pi} = 2\theta r^2.$$

Area of a lune on the unit sphere: 2θ, where θ is the "lunar angle".

Girand's formula



The area for a spherical triangle ABC is given by

$$\textbf{Girand's formula:} \quad \alpha + \beta + \gamma - \pi.$$

For a sphere of radius r:

$$r^2(\alpha + \beta + \gamma - \pi).$$

Proof of Girand's formula

Consider the points of intersection of great circles of a triangle. Mark A' antipodal to A, B' antipodal to B and C' antipodal to C.

Let Δ be the area of ABC and let x, y and z be the areas of $A'BC$, $AB'C$ and ABC' respectively.

Δ and x together make up the area of a lune whose angle is α. Similar facts apply to Δ, y and to Δ, z. So we have

$$\Delta + x = 2\alpha$$
$$\Delta + y = 2\beta$$
$$\Delta + z = 2\gamma$$

Adding,

$$3\Delta + x + y + z = 2(\alpha + \beta + \gamma).$$

Now look at the spherical triangle with vertices A, B', C'. These points are antipodal to the points A', B, C and so these two triangles have the same area, x.

Thus, the great circle through $C'; B; C; B'$ defines a hemisphere whose area is $\Delta + x + y + z$. Therefore

$$\Delta + x + y + z = 2\pi$$

Substituting $x + y + z = 2\pi - \Delta$ into our previous equation, we get

$$\Delta = \alpha + \beta + \gamma - \pi.$$

Corollary 1:
It follows from Girard's formula that the sum of the angles in a spherical triangle must be at least π.

Contrast this with Euclidean Geometry (angle sum $= \pi$) and Hyperbolic Geometry (angle sum $< \pi$.)

Corollary 2:
If two spherical triangles are similar, then they have the same area.