Distance between two points on the Earth

$\mathbf{X}, \mathbf{Y}$: **unit vectors** pointing in the directions of the two points.

We can compute the dot product easily:

$$\mathbf{X} \cdot \mathbf{Y} = x_1 y_1 + x_2 y_2 + x_3 y_3$$

And we know that

$$\mathbf{X} \cdot \mathbf{Y} = |\mathbf{X}||\mathbf{Y}| \cos(\alpha) = \cos(\alpha) (|\mathbf{X}| = |\mathbf{Y}| = 1)$$

where α is the angle between the two vectors.

Finally the length of the arc of the great circle joining the two points is given by $R\alpha$, where R is the radius.

NB: α must be in radians.

Example

Find the length of the shortest path between Prague and Winnipeg. The unit vectors corresponding to these two locations are

$$\mathbf{X} = (0.6214, 0.1597, 0.7669), \mathbf{Y} = (-0.0795, -0.6389, 0.7651).$$

Let α be the angle between the vectors $\mathbf{X}$ and $\mathbf{Y}$. We use the dot product:

$$\cos(\alpha) = \mathbf{X} \cdot \mathbf{Y}$$

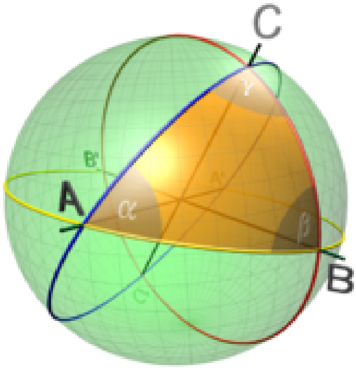
since $\mathbf{X}$ and $\mathbf{Y}$ are unit vectors. Then

$$\cos(\alpha) = \mathbf{X} \cdot \mathbf{Y} = 0.4352 \quad \text{so} \quad \alpha = \cos^{-1}(0.4352) = 1.1204$$

This is the distance between $\mathbf{X}$ and $\mathbf{Y}$ on the unit sphere. Multiply by the earth's radius, 6377.5 km and we find that the shortest distance between Prague and Winnipeg is 7145.641 km.
(Google gives 7139 km.)

Spherical triangles

Three great circles define a spherical triangle.



Spherical Angles

The angle between two great circles is defined to be the angle between the planes containing them.

The angle between two intersecting planes is equal to the angle between their normal vectors. This angle can be found using scalar products.

(Recall: the **normal** is a vector perpendicular to the plane.)

Normal vectors can be found using vector products.

Example

Find the angle between the plane P_1 containing the vectors $(0, 1, 1)$ and $(1, 2, -1)$ and the plane P_2 containing the vectors $(1, 0, 1)$ and $(2, 1, 3)$.

The sides and angles of a spherical triangle on the **unit sphere**

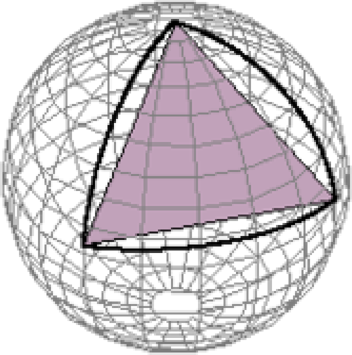
Each side is measured by the angle at the centre of the sphere (radian measure!)

Why? The length of an arc is $R\alpha$ where R is the radius of the great circle or sphere, and $R = 1$ on a unit sphere.

The angles are the angles between the planes containing the two great circles that define the angle.

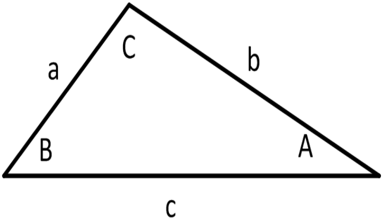
Spherical triangles

The angles in a spherical triangle are bigger than the angles in the plane triangle.



It is possible to make a spherical triangle in which every angle is $90°$ or more.

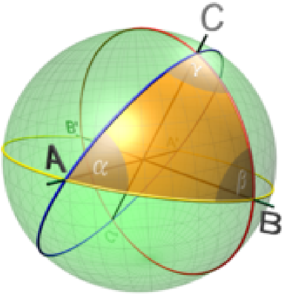
The Sine and Cosine Rules for Plane Triangles



Cosine Rule: $a^2 = b^2 + c^2 - 2bc\cos(A)$

Sine Rule: $\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}$.

The cosine rule for spherical triangles



Consider the angle α. This is the angle between the planes defined by OAB and OAC. Let $a = \angle BOC$, $b = \angle AOC$ and $c = \angle AOB$

Recall that the angle between two vectors $\mathbf{X}, \mathbf{Y}$ is given by

$$\cos(\alpha) = \frac{\mathbf{X} \cdot \mathbf{Y}}{|\mathbf{X}||\mathbf{Y}|}.$$

The angle between two planes is the same as the angle between vectors that are normal (perpendicular) to the planes.

So we take the normal vectors $\mathbf{A} \times \mathbf{B}$ and $\mathbf{A} \times \mathbf{C}$, to get

$$\cos(\alpha) = \frac{(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{A} \times \mathbf{C})}{|\mathbf{A} \times \mathbf{B}||\mathbf{A} \times \mathbf{C}|}$$

Since $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are unit vectors, $|\mathbf{A} \times \mathbf{B}| = \sin(c)$ and $|\mathbf{A} \times \mathbf{C}| = \sin(b)$. So

$$\cos(\alpha) = \frac{(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{A} \times \mathbf{C})}{\sin(c)\sin(b)}$$

Next, we use

$$\mathbf{X} \cdot (\mathbf{Y} \times \mathbf{Z}) = \mathbf{Y} \cdot (\mathbf{Z} \times \mathbf{X}).$$

This gives

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{A} \times \mathbf{C}) = \mathbf{B} \cdot ((\mathbf{A} \times \mathbf{C}) \times \mathbf{A})$$

(Now use the formula for the triple vector product)

$$= \mathbf{B} \cdot ((\mathbf{A} \cdot \mathbf{A})\mathbf{C} - (\mathbf{C} \cdot \mathbf{A})\mathbf{A})$$

$\mathbf{A}$ is a unit vector, so $\mathbf{A} \cdot \mathbf{A} = 1$.

$$= \mathbf{B} \cdot \mathbf{C} - (\mathbf{C} \cdot \mathbf{A})(\mathbf{A} \cdot \mathbf{B})$$

$$= \cos(a) - \cos(b)\cos(c).$$

So we have

$$\cos(\alpha) = \frac{\cos(a) - \cos(b)\cos(c)}{\sin(c)\sin(b)}$$

This gives us the **spherical cosine rule**:

$$\cos(a) = \cos(\alpha)\sin(b)\sin(c) + \cos(b)\cos(c).$$