Dot and Cross Products in $\mathbb{R}^3$

Dot product:

$$\mathbf{X} \cdot \mathbf{Y} = x_1 y_1 + x_2 y_2 + x_3 y_3$$

$$\mathbf{X} \cdot \mathbf{Y} = |\mathbf{X}||\mathbf{Y}| \cos(\theta)$$

$$\mathbf{X} \perp \mathbf{Y} \iff \mathbf{X} \cdot \mathbf{Y} = 0.$$

Cross product:

$$|\mathbf{X} \times \mathbf{Y}| = |\mathbf{X}||\mathbf{Y}| \sin(\theta).$$

The direction is chosen so that $\mathbf{X} \times \mathbf{Y}$ is orthogonal to the plane containing $\mathbf{X}$ and $\mathbf{Y}$ and so that the three vectors $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{X} \times \mathbf{Y}$ form a 'right-handed system'.

$$\mathbf{X} \times \mathbf{Y} = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1).$$

The Scalar Triple Product

Let **A**, **B**, and **C** be three vectors in $\mathbb{R}^3$. The **scalar triple product** is

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$$

If $\mathbf{A} = (a_1, a_2, a_3)$, $\mathbf{B} = (b_1, b_2, b_3)$ and $\mathbf{C} = (c_1, c_2, c_3)$, then

$$\begin{aligned} \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) &= (a_1, a_2, a_3) \cdot (b_2c_3 - b_3c_2, b_3c_1 - b_1c_3, b_1c_2 - b_2c_1) \\ &= (a_1(b_2c_3 - b_3c_2)) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)) \\ &= (a_1(b_2c_3 - b_3c_2)) - a_2(b_1c_3 - b_3c_3) + a_3(b_1c_2 - b_2c_1)) \end{aligned}$$

This is the determinant of the 3×3 matrix that has the components of **A**, **B**, and **C** as its rows.

Properties of the Scalar Triple Product

1. The scalar triple product is given by a determinant:

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

2. It follows from properties of determinants that

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A})$$

3. $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = 0$ if $\mathbf{A}$ lies in the same plane as $\mathbf{B}$ and $\mathbf{C}$.
4. If any two of the vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are the same, then $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = 0$.

The Vector Triple Product

This is the vector $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$. Warning: the vector product is not associative. For example,

$$\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

but

$$(\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = \mathbf{0} \times \mathbf{j} = \mathbf{0}.$$

How can we picture the Vector Triple Product?

$\mathbf{B} \times \mathbf{C}$ is perpendicular to both vectors. So it is on the line through the origin that is perpendicular to the plane containing $\mathbf{B}$ and $\mathbf{C}$.

So $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ is back in the plane containing $\mathbf{B}$ and $\mathbf{C}$ and is perpendicular to $\mathbf{A}$.

If $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{0}$, but $\mathbf{B} \times \mathbf{C} \neq \mathbf{0}$, then $\mathbf{A}$ is perpendicular to the plane containing $\mathbf{B}$ and $\mathbf{C}$.

A formula for the vector triple product:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$$

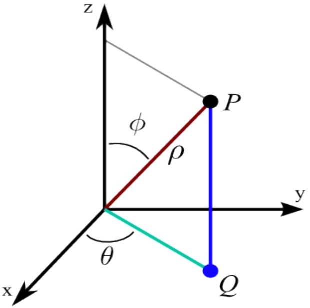
Proof:

Examples

Let $\mathbf{A} = (1, 0, 2)$, $\mathbf{B} = (1, -2, 3)$ and $\mathbf{C} = (-1, 1, 2)$.

Find $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$, $\mathbf{B} \cdot (\mathbf{A} \times \mathbf{C})$, $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ and $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$.

Spherical coordinates



$$x = \rho\cos(\theta)\sin(\phi)$$
$$y = \rho\sin(\theta)\sin(\phi)$$
$$z = \rho\cos(\phi).$$

Example

Find the spherical coordinates and the cartesian coordinates of

Prague $50°05'$N, $14°25'$E

Winnipeg $49°55'$N, $97°06'$W.

Take the radius of the Earth to be 6377.5 kilometres.